# ECE801
# Monitoring and Estimation

# **Kalman Filter**

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# Outline

- 1-D Kalman Filter

# Vehicle Tracking Problem


- Assume a vehicle that moves on a straight line with constant velocity $v$.
- State variable is the position $x$ and the dynamics of the system are given by

$$\dot{x} = \frac{dx}{dt} = v \qquad \text{with initial condition } x(0) = x_0$$

- So

$$x(t) = \int_0^t v \, d\tau = x_0 + vt$$

- But, often there are disturbances

# Vehicle Tracking Problem


- In discrete time,

$$\frac{x_{k+1} - x_k}{\Delta} = v$$

- Where $x_k = x(k\Delta)$ and $\Delta$ is the sampling interval
- Thus

$$x_{k+1} = x_k + v\Delta \qquad\qquad x_{k+1} = x_0 + kv\Delta$$

- Again, what if there are disturbances and velocity is not constant?

# Vehicle Tracking Problem

- Assume that every Δ seconds we get a (noisy) measurement of the position,

$$y_k = x_k + w_k$$

  where $w_k$ is the noise..

- If for some $k$ the noise $w_k$ is large, then our sensor measurement will be of low quality and will contain significant error.
- Can we use available information to reduce the sensor measurement error?
  - What information?
  - How?

# Vehicle Tracking Problem

- **What information**?
- Velocity is constant.
  - The velocity should not change significantly from one time instant to the next!
- We have an estimate of where the vehicle was up to the previous time instant $k-1$, thus assuming fast enough sampling, the vehicle it cannot be “very far” from the previous position.
- **Notation**
  - $\hat{x}_{k,k-1}$ Predicted position **before** measurement $k$ is received
  - $\hat{x}_{k,k}$ Estimated position **after** measurement $k$ is consider
  - $\hat{v}_{k,k-1}$ Predicted velocity **before** measurement $k$ is received
  - $\hat{v}_{k,k}$ Estimated velocity **after** measurement $k$ is considered

# Vehicle Tracking Problem

- At time $k-1$
  - We have the predicted position and velocity $\hat{x}_{k-1,k-1}$ and $\hat{v}_{k-1,k-1}$
- Use the model to **predict** the position and velocity of the vehicle
  - Position is **predicted** through the model

$$\hat{x}_{k,k-1} = \hat{x}_{k-1,k-1} + \Delta \hat{v}_{k-1,k-1}$$

  - Velocity is constant, thus

$$\hat{v}_{k,k-1} = \hat{v}_{k-1,k-1}$$

- Then use the new measurement to “correct” the predicted values

# Vehicle Tracking Problem

- At time $k$
  - We have the predicted position and velocity $\hat{x}_{k,k-1}$ and $\hat{v}_{k,k-1}$
  - And a new measurement of the position $y_n$
- Recall the recursive sample average

$$\bar{x}_{k+1} = \bar{x}_k + \frac{1}{n+1}\left(x_{k+1} - \bar{x}_k\right)$$

Weights $\alpha$ and $\beta$ depend on our confidence on the sensor measurement

  - Position update

$$\hat{x}_{k,k} = \hat{x}_{k,k-1} + \alpha\left(y_k - \hat{x}_{k,k-1}\right)$$

  - Velocity update

$$\hat{v}_{k,k} = \hat{v}_{k,k-1} + \frac{\beta}{\Delta}\left(y_k - \hat{x}_{k,k-1}\right)$$

tracking.m

# Vehicle Tracking Problem

- In vector form…
  - Let $x_1 = x$ and $x_2 = v$
  - Then the model in vector form is
  - $$X(k+1) = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = AX(k)$$
  - The Measurement

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} X(k) + w(k) = CX(k) + w(k)$$

# Vehicle Tracking Problem

- Let $\tilde{X}_k$ be the predicted state before measurement $k$ is considered and $\hat{X}_k$ be the estimated position after the $k$th measurement, then

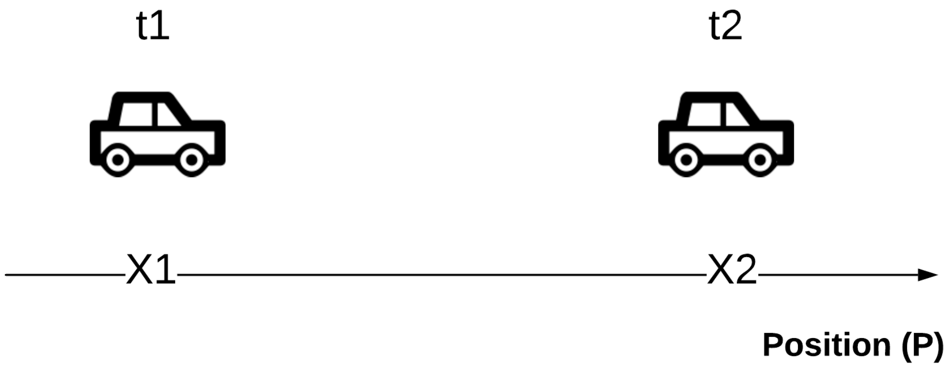
$$\tilde{X}\left(k+1\right)=\begin{bmatrix} \tilde{x}_1(k+1) \\ \tilde{x}_2(k+1) \end{bmatrix}=\begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix}=A\hat{X}(k)$$

and

$$\hat{X}\left(k\right)=\begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix}=\begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{bmatrix}+\begin{bmatrix} \alpha \\ \beta / \Delta \end{bmatrix}\left(y(k)-C\begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{bmatrix}\right)$$

$$=\tilde{X}(k)+K\left(y(k)-C\tilde{X}(k)\right)$$

# Vehicle Tracking: Constant Acceleration




- Assume a vehicle that moves on a straight line with constant acceleration $a$.
- In this case, the state variables are the position $x$ the velocity $v$ and the acceleration $a$ and the dynamics are given by

$$\dot{x} = \frac{dx}{dt} = v \quad \text{with initial condition } x(0) = x_0$$

$$\dot{v} = \frac{dv}{dt} = \alpha \quad \text{with initial condition } v(0) = v_0$$

# Vehicle Tracking: Constant Acceleration

- In discrete time,

$$\frac{x_{k+1} - x_k}{\Delta} = v \qquad \frac{v_{k+1} - v_k}{\Delta} = \alpha$$

- Where $x_k = x(k\Delta)$ and $\Delta$ is the sampling interval
- Thus, the model in vector form

$$X(k+1) = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} x(k+1) \\ v(k+1) \\ a(k+1) \end{bmatrix} = \begin{bmatrix} 1 & \Delta & \Delta^2/2 \\ 0 & 1 & \Delta \\ 0 & 0 & 1 \end{bmatrix} X(k)$$

# Vehicle Tracking: Constant Acceleration

- Assume that every Δ seconds we get a (noisy) measurement of the position,

$$y_k = x_k + w_k = CX_k + w_k$$

where $C = [1\ 0\ 0]$, and $w_k$ is the noise.

- Let $\tilde{X}_k$ be the predicted state before measurement $k$ is considered and $\hat{X}_k$ be the estimated position after the $k$th measurement, then

# Vehicle Tracking: Constant Acceleration

$$\tilde{X}\left(k+1\right)=\begin{bmatrix}\tilde{x}_1(k+1)\\ \tilde{x}_2(k+1)\\ \tilde{x}_3(k+1)\end{bmatrix}=\begin{bmatrix}1 & \Delta & \Delta^2/2\\ 0 & 1 & \Delta\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{x}_1(k)\\ \hat{x}_2(k)\\ \hat{x}_3(k)\end{bmatrix}=A\hat{X}(k)$$

$$\hat{X}\left(k\right)=\begin{bmatrix}\hat{x}_1(k)\\ \hat{x}_2(k)\\ \hat{x}_3(k)\end{bmatrix}=\begin{bmatrix}\tilde{x}_1(k)\\ \tilde{x}_2(k)\\ \tilde{x}_3(k)\end{bmatrix}+\begin{bmatrix}\alpha\\ \beta/\Delta\\ \gamma/\Delta^2\end{bmatrix}\left(y(k)-C\begin{bmatrix}\tilde{x}_1(k)\\ \tilde{x}_2(k)\\ \tilde{x}_3(k)\end{bmatrix}\right)$$

$$=\tilde{X}(k)+K\left(y(k)-C\tilde{X}(k)\right)$$

tracking2.m

# Kalman Gain

- The state estimate update included the Kalman Gain $K$ and we assumed it to be fixed and somehow given to us
- How can we determine the value of the Kalman gain?
- Is it constant, or does it also depend on the data?

Weight that depends on the confidence we give to the previous estimate $\bar{x}_k$ and the latest measurement

- Recall the sample mean again

$$\bar{x}_{k+1} = \bar{x}_k + \frac{1}{k+1}\left(x_{k+1} - \bar{x}_k\right) = \bar{x}_k + K_k\left(x_{k+1} - \bar{x}_k\right)$$

- Let $r_k$ be the uncertainty in the latest measurement $x_{k+1}$
- Let $P_{k,k-1}$ be the (estimated) uncertainty of the $\bar{x}_k$

$$0 \le K_k \le 1$$

# Kalman Gain

- Let us use the previous notation that we had established for the Kalman Filter
  - $y_k$ Latest measurement
  - $\hat{x}_{k,k-1}$ Predicted state **before** measurement $k$ is received
  - $\hat{x}_{k,k}$ Estimated state **after** measurement $k$ is consider
  - $r_k$ is the uncertainty in the latest measurement $y_k$
  - $P_{k,k-1}$ be the (estimated) uncertainty of the $\hat{x}_{k,k-1}$
  - $P_{k,k}$ be the (estimated) uncertainty of the $\hat{x}_{k,k}$

$$\hat{x}_{k,k} = \hat{x}_{k,k-1} + K_k\left(y_k - \hat{x}_{k,k-1}\right)$$

$$K_k = \frac{P_{k,k-1}}{P_{k,k-1} + r_k}$$

Uncertainty of the prior estimate

Uncertainty (variance) of the measurement

# Kalman Gain

- The Kalman gain determines the weight of a new measurement

$$\hat{x}_{k,k} = \hat{x}_{k,k-1} + K_k\left(y_k - \hat{x}_{k,k-1}\right) = (1 - K_k)\hat{x}_{k,k-1} + K_k y_k \qquad 0 \le K_n \le 1$$

- For the sample average let us assume i.i.d. random numbers with some mean $\mu$ and variance $\sigma^2$
- Let $r_k = \sigma^2$ while the variance of the estimate using $k-1$ samples is $P_{k,k-1} = \frac{1}{n-1}\sigma^2$.
- Therefore,

$$K_n = \frac{P_{k,k-1}}{P_{k,k-1} + r_k} = \frac{\frac{1}{n-1}\sigma^2}{\frac{1}{n-1}\sigma^2 + \sigma^2} = \frac{1}{n}$$

Which is the gain used in the recursive sample mean estimate

# Kalman Gain

- How does the variance of the estimate change?
- The variance of the estimate using $k-1$ samples is $P_{k,k-1} = \frac{1}{k-1}\sigma^2$
- The variance after $k$ samples is $P_{k,k} = \frac{1}{k}\sigma^2$
- Therefore,

$$P_{k,k} = (1-K_k)P_{k,k-1} = \left(1-\frac{1}{k}\right)\frac{\sigma^2}{k-1} = \frac{\sigma^2}{k}$$

- Also, before the measurement is considered,

$$P_{k,k-1} = P_{k-1,k-1}$$

# Kalman Filter Algorithm

Due to the dynamics or it is 0 if constant parameter

- **Initialize**: $\hat{x}_{0,0}$, $P_{0,0} = r = \sigma^2$
- State prediction $\hat{x}_{k,k-1} = \hat{x}_{k-1,k-1} + f_{k-1}$
- Covariance prediction $P_{k,k-1} = P_{k-1,k-1}$
- Kalman Gain Computation $K_k = \dfrac{P_{k,k-1}}{P_{k,k-1} + r_k}$
- State Update $\hat{x}_{k,k} = \hat{x}_{k,k-1} + K_k\left(y_k - \hat{x}_{k,k-1}\right)$
- Covariance Update $P_{k,k} = (1 - K_k)P_{k,k-1}$

# Constant speed vehicle tracking problem

- Recall the state $x_1 = x$ and $x_2 = v$ and

$$X(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k-1) \\ x_2(k-1) \end{bmatrix} = AX(k-1)$$

- Thus the state prediction $\hat{X}_{k,k-1} = A\hat{X}_{k-1,k-1}$
- The variance of the position $p_{k,k-1} = p_{k-1,k-1}$

No randomness involved

- Kalman Gain Computation $K_k = \dfrac{p_{k,k-1}}{p_{k,k-1} + r_k}$
- Position Update $\hat{x}_{k,k} = \hat{x}_{k,k-1} + K_k\left(y_k - \hat{x}_{k,k-1}\right)$
- Position variance Update $p_{k,k} = (1 - K_k)p_{k,k-1}$

# Vehicle Tracking Problem

- In vector form…
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  - The Measurement

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} X(k) + w(k) = CX(k) + w(k)$$

# Constant speed vehicle tracking problem

Due to the dynamics or it is 0 if constant parameter

- **Initialize**: $\hat{x}_{0,0}$, $P_{0,0} = r = \sigma^2$
- State prediction $$\hat{x}_{k,k-1} = \hat{x}_{k-1,k-1} + f_{k-1}$$
- Covariance prediction $$P_{k,k-1} = P_{k-1,k-1}$$
- Kalman Gain Computation $$K_k = \frac{P_{k,k-1}}{P_{k,k-1} + r_k}$$
- State Update $$\hat{x}_{k,k} = \hat{x}_{k,k-1} + K_k\left(y_k - \hat{x}_{k,k-1}\right)$$
- Covariance Update $$P_{k,k} = (1 - K_k)P_{k,k-1}$$