Suppose we want to multiply the following two polynomials

$A(x)=2\mathrm{x}+3, \qquad B(x)=1\mathrm{x}+1$

The degree of each polynomial is (n-1)=1, thus n=2.

Brute force approach:

This can be done using brute force. The result is given by

$$A(x)*B(x)=(2\mathrm{x}+3)*(x+1)$$
$$=2*1\,x^2+2*1\,x+3*1\,x+3*1$$
$$=2\mathrm{x}^2+5\mathrm{x}+3$$

Notice that the number of multiplications required for this approach is of the order $\Theta(n^2)$.

The FFT approach

It is convenient to compute the value of the polynomials at the complex roots of the equation $x^{2\mathrm{n}}=1$. This equation has roots at points

$$w_i=e^{j\frac{2\pi \mathrm{i}}{2\mathrm{n}}} \text{ for all } \quad i=0,1,\cdots,2\mathrm{n}-1.$$

For the above example, n=2, therefore the required complex points are $w_0=1,\ w_1=j,\ w_2=-1,\ w_3=-j.$ Therefore we can compute both polynomials at these points as follows

$$\begin{array}{lclcl} A(w_0)=A(1)=5 & & B(w_0)=B(1)=2 & & C_0=C(w_0)=A(w_0)*B(w_0)=10 \\ A(w_1)=A(j)=3+2\mathrm{j} & \text{and} & B(w_1)=B(j)=1+j & \text{thus} & C_1=C(w_1)=A(w_1)*B(w_1)=1+5\mathrm{j} \\ A(w_2)=A(-1)=1 & & B(w_2)=B(-1)=0 & & C_2=C(w_2)=A(w_2)*B(w_2)=0 \\ A(w_3)=A(-j)=3-2\mathrm{j} & & B(w_3)=B(-j)=1-j & & C_3=C(w_3)=A(w_3)*B(w_3)=1-5\mathrm{j} \end{array}$$

This operation requires $2n$ multiplications $\Theta(n)$.

Now, we have the value of the resulting polynomial at four points, and we need to interpolate in order to obtain the required coefficients $c_i,\ i=0,\cdots,2\mathrm{n}-2.$ To do so, we form the polynomial

$$D(y)=\sum_{m=0}^{2\mathrm{n}-1} C_m y^{m.}$$

For the above example

$$D(y)=\sum_{m=0}^{3} C_m y^m=10+(1+5\mathrm{j})\,x+0\mathrm{x}^2+(1-5\mathrm{j})\,x^{3.}$$

Next, we need to evaluate the values of $D(y)$ again at the roots of the equation $x^{2\mathrm{n}}=1$, i.e., at points $w_i, i=0,\cdots,2\mathrm{n}-1.$ For the above example, these points are again $w_0=1,\ w_1=j,\ w_2=-1,\ w_3=-j.$

$$D(w_0)=D(1)=10+(1+5\mathrm{j})+(1-5\mathrm{j})=12$$
$$D(w_1)=D(j)=10+j(1+5\mathrm{j})-j(1-5\mathrm{j})=0$$
$$D(w_2)=D(-1)=10-(1+5\mathrm{j})-(1-5\mathrm{j})=8$$
$$D(w_3)=D(-j)=10-j(1+5\mathrm{j})+j(1-5\mathrm{j})=20.$$

The last step is to obtain the coefficients of the resulting polynomial using

$$c_0=\frac{1}{2\mathrm{n}}D(w_0), \quad c_{2\mathrm{n}-i}=\frac{1}{2\mathrm{n}}D(w_i), \quad i=2\mathrm{n}-1,\ldots,2$$

For our example

$$c_0 = \frac{1}{4} D(w_0) = \frac{12}{4} = 3, \quad c_1 = \frac{1}{4} D(w_3) = \frac{20}{4} = 5, \quad c_2 = \frac{1}{4} D(w_2) = \frac{8}{4} = 2.$$

Therefore

$$C(x) \;=\; c_0 + c_1 x + c_2 x^2 \;=\; 3 + 5\mathrm{x} + 2\mathrm{x}^2$$

which is precisely the same result that we have obtained using brute force!

Finally, let us make an example where we can use divide and conquer algorithm in order to compute the value of the polynomial. Assume that we want to evaluate *D*(*y*) at the points $w_0 = 1,\ w_1 = j,\ w_2 = -1,\ w_3 = -j.$ We can break *D*(*y*) into odd and event as follows:
$D(y) = D_e(y^2) + y\, D_o(y^2)$ where $D_e(y) = 10 + 0\mathrm{y}$ $\quad D_o(y) = (1+5\mathrm{j}) + (1-5\mathrm{j})\, y$

We need to compute the polynomial at values $w_0 = 1,\ w_1 = j,\ w_2 = -1,\ w_3 = -j.$ The squares of these values are $s_0 = 1,\ s_1 = -1.$ Therefore,

$D_e(s_0) = D_e(1) = 10, \qquad D_e(s_1) = D_e(-1) = 10.$
$D_o(s_0) = D_o(1) = 2, \qquad D_o(s_1) = D_o(-1) = 10\mathrm{j}.$

Finally, we can recombine the result as follows:

$D(w_0) = D_e(s_0) + w_0\, D_o(s_0) = 10 + 1 * 2 = 12$
$D(w_1) = D_e(s_1) + w_1\, D_o(s_1) = 10 + j * 10\mathrm{j} = 0$
$D(w_2) = D_e(s_0) + w_2\, D_o(s_0) = 10 - 1 * 2 = 8$
$D(w_3) = D_e(s_1) + w_3\, D_o(s_1) = 10 - j * 10\mathrm{j} = 20.$