Generalized Semi-Markov Processes (GSMP)

Summary

- Some Definitions
- The Poisson Process
- Properties of the Poisson Process
  - Interarrival times
  - Memoryless property and the residual lifetime paradox
  - Superposition of Poisson processes
- Markov and Semi-Markov Processes

Probability Space and Random Variables

- Probability Space: (Ω, F, P)
  - Ω: **Sample space**: All possible outcomes of a random experiment
  - F: **Event space:** A collection of subsets of Ω for which the following properties should hold.
    - If an event $A \in F$, then the complement of A, $A^c \in F$
    - If two or more events are in F, then their union should also be in F.
  - P: **Probability:** A mapping from the event space to a real number [0,1] which describes the likelihood that the event will occur.
- Random Variable $X(\omega)$
  - A mapping from Ω to the set of real numbers
- Example random experiment: toss a coin
  - Ω: {H, T}, F: $\{\varnothing, H, T, \Omega\}$, P: $P[H]$, $P[T]$
  - $X(T)= 0$, $X(H)= 1$

Random Process

- Let (Ω, F, P) be a probability space. A stochastic (or random) process $\{X(t)\}$ is a collection of random variables defined on (Ω, F, P), indexed by $t \in T$ (where t is usually time). $X(t)$ is the state of the process.
- Continuous Time and Discrete Time stochastic processes
  - If the set T is finite or countable then $\{X(t)\}$ is called *discrete-time process*. In this case $t \in \{0, 1, 2,\ldots\}$ and we may referred to a stochastic sequence. We may also use the notation $\{X_k\}$, $k=0,1,2,\ldots$
  - Otherwise, the process is called *continuous-time* process
- Continuous State and Discrete State stochastic processes
  - If $\{X(t)\}$ is defined over a countable set, then the process is discrete-state, also referred to as chain.
  - Otherwise, the process is continuous-state.

Classification of Random Processes

- Joint cdf of the random variables $X(t_0),\ldots,X(t_n)$

$$F_X\left(x_0,\ldots,x_n;t_0,\ldots,t_n\right)=\Pr\left\{X\left(t_0\right)\le x_0,\ldots,X\left(t_n\right)\le x_n\right\}$$

- Independent Process
  - Let $X_1,\ldots,X_n$ be a sequence of independent random variables, then

$$F_X\left(x_0,\ldots,x_n;t_0,\ldots,t_n\right)=F_{X_0}\left(x_0;t_0\right)\times\ldots\times F_{X_n}\left(x_n;t_n\right)$$

- Stationary Process (strict sense stationarity)
  - The sequence $\{X_n\}$ is stationary if and only if for any $\tau\in R$

$$F_X\left(x_0,\ldots,x_n;t_0+\tau,\ldots,t_n+\tau\right)=F_X\left(x_0,\ldots,x_n;t_0,\ldots,t_n\right)$$

Classification of Random Processes

- Wide-sense Stationarity
  - Let C be a constant and $g(\tau)$ a function of τ but not of t, then a process is wide-sense stationary if and only if

$$\mathrm{E}\left\{X(t)\right\}=C \quad \text{and} \quad \mathrm{E}\left\{X(t)\cdot X(t+\tau)\right\}=g(\tau)$$

- Markov Process
  - The future of a process does not depend on its past, only on its present

$$\Pr\left\{X\left(t_{k+1}\right)\le x_{k+1}\mid X\left(t_k\right)=x_k,\ldots,X\left(t_0\right)=x_0\right\}$$
$$=\Pr\left\{X\left(t_{k+1}\right)\le x_{k+1}\mid X\left(t_k\right)=x_k\right\}$$

- Also referred to as the Markov property

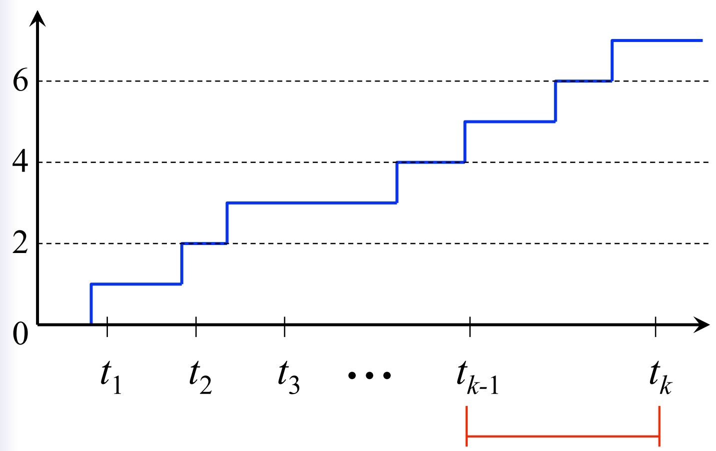
Renewal Process

- A renewal process is a chain $\{N(t)\}$ with state space $\{0,1,2,\ldots\}$ whose purpose is to count state transitions. The time intervals between state transitions are assumed iid from an arbitrary distribution. Therefore, for any $0 \le t_1 \le \ldots \le t_k \le \ldots$

$$N(0) = 0 \le N(t_1) \le N(t_2) \le \ldots \le N(t_k) \le \ldots$$

The Poisson Counting Process

- Let the process $\{N(t)\}$ which counts the number of events that have occurred in the interval $[0,t)$. For any $0 \le t_1 \le \ldots \le t_k \le \ldots$ $N(0) = 0 \le N(t_1) \le N(t_2) \le \ldots \le N(t_k) \le \ldots$



$$N(t_{k-1}, t_k) = N(t_k) - N(t_{k-1})$$

- Process with independent increments: The random variables $N(t_1)$, $N(t_1,t_2),\ldots,$ $N(t_{k-1},t_k),\ldots$ are mutually independent.
- Process with stationary independent increments: The random variable $N(t_{k-1}, t_k)$, does not depend on t_{k-1}, t_k but only on $t_k - t_{k-1}$

The Poisson Counting Process

Assumptions:

- At most one event can occur at any time instant (no two or more events can occur at the same time)
- A process with stationary independent increments

$$\Pr\{N(t_{k-1},t_k)=n\}=\Pr\{N(t_k-t_{k-1})=n\}$$

- Given that a process satisfies the above assumptions, find

$$P_n(t)\equiv\Pr\{N(t)=n\},n=0,1,2,\ldots$$

The Poisson Process

- **Step 1**: Determine

$$P_0(t)\equiv\Pr\{N(t)=0\}$$

- Starting from

$$\Pr\{N(t+s)=0\}=\Pr\{N(t)=0\ \text{ and }\ N(t,t+s)=0\}$$

Stationary independent increments

$$=\Pr\{N(t)=0\}\Pr\{N(s)=0\}$$

$$\Rightarrow P_0(t+s)=P_0(t)P_0(s)$$

- **Lemma:** Let $g(t)$ be a differentiable function for all $t\geq 0$ such that $g(0)=1$ and $g(t)\leq 1$ for all $t>0$. Then for any t, $s\geq 0$

$$g(t+s)=g(t)g(s)\Leftrightarrow g(t)=e^{-\lambda t}\quad \text{for some } \lambda>0$$

The Poisson Process

- Therefore $\boxed{P_0(t) \equiv \Pr\{N(t)=0\} = e^{-\lambda t}}$
- **Step 2**: Determine $P_0(\Delta t)$ for a small Δt.

$$\Pr\{N(\Delta t)=0\} = e^{-\lambda\Delta t} = 1-\lambda\Delta t + \frac{(\lambda\Delta t)^2}{2!} - \frac{(\lambda\Delta t)^3}{3!} + \ldots$$
$$= 1-\lambda\Delta t + o(\Delta t).$$

- **Step 3**: Determine $P_n(\Delta t)$ for a small Δt.
- For n=2,3,… since by assumption no two events can occur at the same time

$$P_n(\Delta t) \equiv \Pr\{N(\Delta t)=n\} = o(\Delta t)$$

- As a result, for n=1

$$P_1(\Delta t) \equiv \Pr\{N(\Delta t)=1\} = \lambda\Delta t + o(\Delta t)$$

The Poisson Process

- **Step 4**: Determine $P_n(t+\Delta t)$ for any n

$$P_n(t+\Delta t) \equiv \Pr\{N(t+\Delta t)=n\} = \sum_{k=0}^{n} P_{n-k}(t)P_k(\Delta t)$$
$$= P_n(t)P_0(\Delta t) + P_{n-1}(t)P_1(\Delta t) + o(\Delta t).$$
$$= [1-\lambda\Delta t + o(\Delta t)]P_n(t) + [\lambda\Delta t + o(\Delta t)]P_{n-1}(t) + o(\Delta t).$$

- Moving terms between sides,

$$\frac{P_n(t+\Delta t)-P_n(t)}{\Delta t} = -\lambda P_n(t) + \lambda P_{n-1}(t) + \frac{o(\Delta t)}{\Delta t}.$$

- Taking the limit as $\Delta t \rightarrow 0$

$$\boxed{\frac{dP_n(t)}{dt} = -\lambda P_n(t) + \lambda P_{n-1}(t)}$$

The Poisson Process

- **Step 5**: Solve the differential equation to obtain

$$\boxed{P_n(t) \equiv \Pr\{N(t)=n\} = \frac{(\lambda t)^n}{n!}e^{-\lambda t}, \quad t \geq 0, \quad n=0,1,2,\ldots}$$

- This expression is known as the Poisson distribution and it full characterizes the stochastic process $\{N(t)\}$ in $[0,t)$ under the assumptions that
  - No two events can occur at exactly the same time, and
  - Independent stationary increments
- You should verify that

$$E[N(t)] = \lambda t \quad \text{and} \quad \mathrm{var}[N(t)] = \lambda t$$

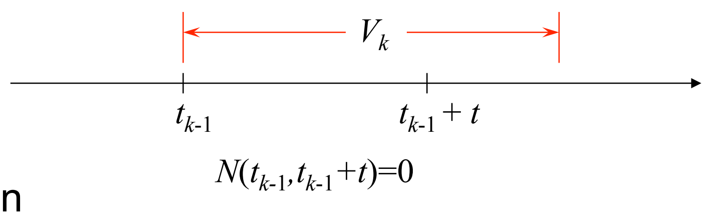
- Parameter λ has the interpretation of the "rate" that events arrive.

Properties of the Poisson Process: Interevent Times

- Let $t_{k\text{-}1}$ be the time when the k-1 event has occurred and let V_k denote the (random variable) interevent time between the kth and k-1 events.
- What is the cdf of V_k, $G_k(t)$?

$$G_k(t) = \Pr\{V_k \leq t\} = 1 - \Pr\{V_k > t\}$$
$$= 1 - \Pr\{0 \text{ arrivals in the interval } [t_{k-1}, t_{k-1}+t)\}$$
$$= 1 - \Pr\{N(t) = 0\}$$
$$= 1 - e^{-\lambda t}$$

Stationary independent increments



$$\boxed{\Rightarrow G(t) = 1 - e^{-\lambda t}}$$

Exponential Distribution

Properties of the Poisson Process: Exponential Interevent Times

- The process $\{V_k\}$ k=1,2,…, that corresponds to the interevent times of a Poisson process is an iid stochastic sequence with cdf

$$G(t) = \Pr\{V_k \le t\} = 1 - e^{-\lambda t}$$

Therefore, the Poisson is also a **renewal** process

- The corresponding pdf is

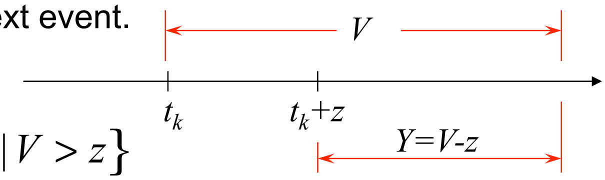
$$g(t) = \lambda e^{-\lambda t}, \quad t \ge 0$$

- One can easily show that

$$E[V_k] = \frac{1}{\lambda} \qquad \text{and} \qquad \mathrm{var}[V_k] = \frac{1}{\lambda^2}$$

Properties of the Poisson Process: Memoryless Property

- Let t_k be the time when the previous event has occurred and let V denote the time until the next event.
- Assuming that we have been at the current state for z time units, let Y be the remaining time until the next event.
- What is the cdf of Y?



$$F_Y(t) = \Pr\{Y \le t\} = \Pr\{V - z < t \mid V > z\}$$

$$= \frac{\Pr\{V > z \text{ and } V < z+t\}}{\Pr\{V > z\}} = \frac{\Pr\{z < V < z+t\}}{1 - \Pr\{V < z\}}$$

$$= \frac{G(t+z) - G(z)}{1 - G(z)} = \frac{1 - e^{-\lambda(t+z)} - 1 + e^{-\lambda z}}{1 - 1 + e^{-\lambda z}}$$

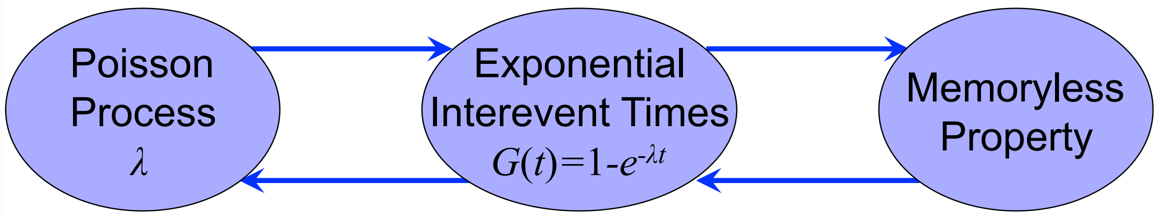
$$\Rightarrow F_Y(t) = 1 - e^{-\lambda t} = G(t)$$

Memoryless! It does not matter that we have already spent z time units at the current state.

Memoryless Property

- This is a unique property of the exponential distribution. If a process has the memoryless property, then it must be exponential, i.e.,

$$\Pr\{V \le z+t \mid V > z\} = \Pr\{V \le t\} \Leftrightarrow \Pr\{V \le t\} = 1 - e^{-\lambda t}$$

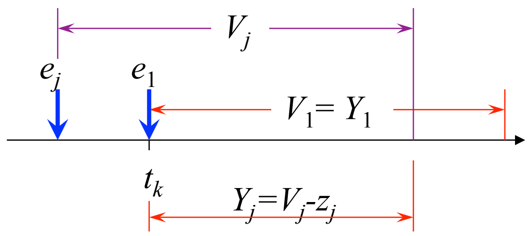


Superposition of Poisson Processes

- Consider a DES with m events each modeled as a Poisson Process with rate λ_i, $i=1,\ldots,m$. What is the resulting process?
- Suppose at time t_k we observe event 1. Let Y_1 be the time until the next event 1. Its cdf is $G_1(t)=1\text{-}\exp\{-\lambda_1 t\}$.
- Let $Y_2,\ldots,Y_m$ denote the residual time until the next occurrence of the corresponding event.
- Their cdfs are:

$$G_i(t) = 1 - e^{-\lambda_i t}$$

Memoryless Property



- Let Y^* be the time until the next event (any type).

$$Y^* = \min\{Y_i\}$$

- Therefore, we need to find

$$G_{Y^*}(t) = \Pr\{Y^* \le t\}$$

Superposition of Poisson Processes

$$G_{Y^*}(t) = \Pr\{Y^* \le t\} = \Pr\{\min\{Y_i\} \le t\}$$

$$= 1 - \Pr\{\min\{Y_i\} > t\}$$

$$= 1 - \Pr\{Y_1 > t, \ldots, Y_m > t\}$$

Independence

$$= 1 - \prod_{i=1}^{m} \Pr\{Y_i > t\} = 1 - \prod_{i=1}^{m} e^{-\lambda_i t}$$

$$\Rightarrow G_{Y^*}(t) = 1 - e^{-\Lambda t} \quad \text{where} \quad \Lambda = \sum_{i=1}^{m} \lambda_i$$

- The superposition of m Poisson processes is also a Poisson process with rate equal to the sum of the rates of the individual processes

Superposition of Poisson Processes

- Suppose that at time t_k an event has occurred. What is the probability that the next event to occur is event j?
- Without loss of generality, let $j=1$ and define

$$Y' = \min\{Y_i : i = 2, \ldots, m\}. \quad \sim 1\text{-}\exp\left\{-\sum_{i=2}^{m} \lambda_i t\right\} = 1 - \exp\{-\Lambda' t\}$$

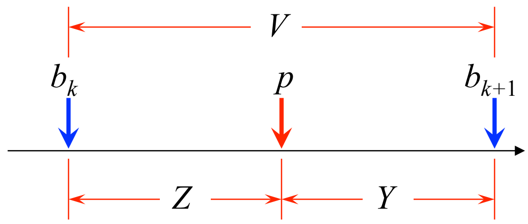
$$\Pr\{\text{next event is } j = 1\} = \Pr\{Y_1 \le Y'\} =$$

$$= \int_0^{\infty} \int_0^{y'} \lambda_1 e^{-\lambda_1 y_1} \Lambda' e^{-\Lambda' y'} dy_1 dy'$$

$$= \int_0^{\infty} \Lambda' (1 - e^{-\lambda_1 y_1}) e^{-\Lambda' y'} dy'$$

$$= \frac{\lambda_1}{\Lambda} \quad \text{where} \quad \Lambda = \sum_{i=1}^{m} \lambda_i$$

Residual Lifetime Paradox



- Suppose that buses pass by the bus station according to a Poisson process with rate λ. A passenger arrives at the bus station at some random point.
- How long does the passenger has to wait?
- **Solution 1**:
  - $E[V] = 1/\lambda$. Therefore, since the passenger will (on average) arrive in the middle of the interval, he has to wait for $E[Y]=E[V]/2= 1/(2\lambda)$.
  - But using the memoryless property, the time until the next bus is exponentially distributed with rate λ, therefore $E[Y]=1/\lambda$ not $1/(2\lambda)$!
- **Solution 2**:
  - Using the memoryless property, the time until the next bus is exponentially distributed with rate λ, therefore $E[Y]=1/\lambda$.
  - But note that $E[Z]= 1/\lambda$ therefore $E[V]= E[Z]+E[Y]= 2/\lambda$ not $1/\lambda$!

Markov and Semi-Markov Property

- The Markov Property requires that the process has no memory of the past. This memoryless property has two aspects:
  - All past state information is irrelevant in determining the future (no state memory).
  - How long the process has been in the current state is also irrelevant (no state age memory).
  - The later implies that the lifetimes between subsequent events (interevent time) should also have the memoryless property (i.e., exponentially distributed).
- Semi-Markov Processes
  - For this class of processes the requirement that the state age is irrelevant is relaxed, therefore, the interevent time is no longer required to be exponentially distributed.

Example

- Consider the process

$$X_{k+1} = X_k - X_{k-1}$$

with $\Pr\{X_0=0\}=\Pr\{X_0=1\}=$ 0.5 and $\Pr\{X_1=0\}=\Pr\{X_1=1\}=$ 0.5

- Is this a Markov process? NO
- Is it possible to make the process Markov?
- Define $Y_k = X_{k-1}$ and form the vector $Z_k = [X_k, Y_k]^T$ then we can write

$$Z_{k+1} = \begin{bmatrix} X_{k+1} \\ Y_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} X_k \\ Y_k \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} Z_k$$

Generalized Semi-Markov Processes (GSMP)

- A GSMP is a stochastic process $\{X(t)\}$ with state space X generated by a stochastic timed automaton

$$(X, E, \Gamma, f, p_0, G)$$

- X is the countable state space
- E is the countable event set
- $\Gamma(x)$ is the feasible event set at state x.
- $f(x, e)$: is state transition function.
- p_0 is the probability mass function of the initial state
- G is a vector with the cdfs of all events.

- The semi-Markov property of GSMP is due to the fact that at the state transition instant, the next state is determined by the current state and the event that just occurred.