Homework 1

1. Markov's Inequality (*): Assume a random variable $X \geq 0$. Show that for any $a > 0$,
$$\Pr\{X \geq a\} \leq \frac{\mathbb{E}[X]}{a}$$

2. Chebyshev's Inequality (*): Assume X is a random variable having mean μ and variance σ^2. Show that for any $k > 0$,
$$\Pr\{|X - \mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

3. Let $X_1, \cdots, X_n$ be a sequence of independent identically distributed random variables with $\mathbb{E}[X_i] = \mu$ and variance σ^2. Answer the following questions

   (a) Find
   $$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \quad \text{and} \quad \text{Var}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right]$$

   (b) Find
   $$\mathbb{E}\left[\frac{a}{n}\sum_{i=1}^{n} X_i + b\right] \quad \text{and} \quad \text{Var}\left[\frac{a}{n}\sum_{i=1}^{n} X_i + b\right]$$
   where a and b are known constants.

   (c) Show that
   $$\Pr\left\{\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| > \epsilon\right\} \to 0 \quad \text{as} \quad n \to \infty$$

4. Find the mean and variance of the following variables

   (a) Binomial random variable with probability mass function
   $$p_i \equiv \Pr\{X = i\} = \binom{n}{i} p^i (1-p)^{n-i}, \ i = 0, 1, \cdots, n$$

   (b) Geometric random variable with probability mass function
   $$p_n \equiv \Pr\{X = n\} = p(1-p)^{n-1}, \ n > 0$$

   (c) Uniform random variable with density
   $$p(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

   (d) Exponential random variable with density
   $$p(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

5. (*) Let $\mathbb{E}[X|Y]$ denote that function of the random variable Y whose value at $Y = y$ is $\mathbb{E}[X|Y = y]$; and note that $\mathbb{E}[X|Y]$ is itself a random variable. Show that
$$\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$$

(*) optional.