Homework 1 Solution

1.

$$\begin{aligned}\mathbb{E}[X] &= \int_0^\infty x f_X(x)dx \\ &= \int_0^a x f_X(x)dx + \int_a^\infty x f_X(x)dx \\ &\geq \int_a^\infty x f_X(x)dx \quad \text{note that } X > 0 \\ &\geq a\int_a^\infty f_X(x)dx^1 \\ &= a\Pr\{X \geq a\}\end{aligned}$$

Therefore,

$$\mathbb{E}[X] \geq a\Pr\{X \geq a\} \Rightarrow \Pr\{X \geq a\} \leq \frac{\mathbb{E}[X]}{a}$$

2. Suppose we define a new random variable $Y = (X - \mu)^2$, thus $Y \geq 0$. Subsequently, applying the Markov inequality assuming $a = k^2\sigma^2$ we get

$$\begin{aligned}\Pr\{Y \geq k^2\sigma^2\} &\leq \frac{\mathbb{E}[Y]}{k^2\sigma^2} \Rightarrow \\ \Pr\{(X-\mu)^2 \geq k^2\sigma^2\} &\leq \frac{\mathbb{E}\left[(X-\mu)^2\right]}{k^2\sigma^2}\end{aligned}$$

By definition $\mathbb{E}\left[(X-\mu)^2\right] = \sigma^2$. Also taking the square root for both sides of the inequality inside the probability, we obtain the required result

$$\Pr\{|X-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

3. (a) Let $Y = \frac{1}{n}\sum_{i=1}^n X_i$, then

$$\mathbb{E}[Y] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n X_i\right] = \frac{1}{n}\sum_{i=1}^n \mathbb{E}[X_i] = \frac{n\mathbb{E}[X_i]}{n} = \mathbb{E}[X_i] = \mu$$

$$
\begin{aligned}
\mathrm{Var}[Y] &= \mathbb{E}\left[(Y-\mu)^2\right] \\
&= \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{n\mu}{n}\right)^2\right] \\
&= \frac{1}{n^2}\mathbb{E}\left[\left(\sum_{i=1}^{n} X_i - n\mu\right)^2\right] \\
&= \frac{1}{n^2}\mathbb{E}\left[\left(\sum_{i=1}^{n} (X_i - \mu)\right)^2\right] \\
&= \frac{1}{n^2}\mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} (X_i - \mu)(X_j-\mu)\right] \\
&= \frac{1}{n^2}\mathbb{E}\left[\sum_{i=1}^{n} (X_i - \mu)^2\right] + \frac{1}{n^2}\mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} (X_i - \mu)(X_j-\mu)\right] \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{E}\left[(X_i - \mu)^2\right] + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\mathbb{E}\left[(X_i - \mu)(X_j-\mu)\right] \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 \\
&= \frac{\sigma^2}{n}
\end{aligned}
$$

(b) Let $Z = aY + b$, where Y is the random variable from (a) above.

$$\mathbb{E}[Z] = \mathbb{E}[aY+b] = a\mathbb{E}[Y] + b = a\mu + b$$

$$
\begin{aligned}
\mathrm{Var}[Z] &= \mathbb{E}\left[(Z-(a\mu+b))^2\right] \\
&= \mathbb{E}\left[(aY+b-a\mu-b)^2\right] \\
&= a^2\mathbb{E}\left[(Y-\mu)^2\right] \\
&= \frac{a^2\sigma^2}{n}
\end{aligned}
$$

(c) Let $Y = \frac{1}{n}\sum_{i=1}^{n} X_i$, then using Chebyshev's inequality we get,

$$\Pr\left\{|Y-\mu| \geq \frac{k\sigma}{\sqrt{n}}\right\} \leq \frac{1}{k^2}$$

Recall that $\mathbb{E}[Y] = \mu$ and $\mathrm{Var}[Y] = \sigma^2/n$. Next, let

$$\epsilon = \frac{k\sigma}{\sqrt{n}} \quad \Rightarrow \quad k = \frac{\sqrt{n}\epsilon}{\sigma}$$

Substituting, in the above equation we get

$$\Pr\left\{\left|\frac{1}{n}\sum_{i=1}^{n} - \mu\right| \geq \epsilon\right\} \leq \frac{\sigma^2}{n\epsilon^2}$$

which clearly goes to 0 as $n \to \infty$.

4. (a) Binomial Distribution: Let $\{Y_i\}$ be the sequence of iid random variables

$$Y_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } (1-p) \end{cases} \qquad i = 1, \cdots, n.$$

Note that $\mathbb{E}[Y] = p$ and $\text{Var}[Y] = (1-p)^2 p + p^2(1-p) = (1-p)p$. Then, we can rewrite X as

$$X = \sum_{i=1}^{n} Y_i$$

Therefore,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} \mathbb{E}[Y_i] = \sum_{i=1}^{n} p = np$$

$$\text{Var}[X] = \text{Var}\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} \text{Var}[Y_i] = n(1-p)p$$

(b) Geometric Distribution: Let $q = 1 - p$,

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{n=1}^{\infty} n p_n = \sum_{n=1}^{\infty} n p (1-p)^{n-1} \\
&= (1-q) \sum_{n=0}^{\infty} \frac{d}{dq} \{q^n\} \\
&= (1-q) \frac{d}{dq} \left\{ \sum_{n=0}^{\infty} q^n \right\} \\
&= (1-q) \frac{d}{dq} \left\{ \frac{1}{1-q} \right\} \\
&= (1-q) \frac{1}{(1-q)^2} \\
&= \frac{1}{1-q} \\
&= \frac{1}{p}
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}\left[X^2\right] &= \sum_{n=1}^{\infty} n^2 p_n = \sum_{n=1}^{\infty} n^2 p(1-p)^{n-1} \\
&= (1-q)\sum_{n=0}^{\infty} \frac{d}{dq}\left\{nq^n\right\} \\
&= (1-q)\sum_{n=0}^{\infty} \frac{d}{dq}\left\{qnq^{n-1}\right\} \\
&= (1-q)\sum_{n=0}^{\infty} \left[nq^{n-1} + q\frac{d}{dq}\left\{nq^{n-1}\right\}\right] \\
&= (1-q)\left[\sum_{n=0}^{\infty} \frac{dq^n}{dq} + q\sum_{n=0}^{\infty} \frac{d^2}{dq^2}\left\{q^n\right\}\right] \\
&= (1-q)\left[\frac{d}{dq}\left\{\sum_{n=0}^{\infty} q^n\right\} + q\frac{d^2}{dq^2}\left\{\sum_{n=0}^{\infty} q^n\right\}\right] \\
&= (1-q)\left[\frac{d}{dq}\left\{\frac{1}{1-q}\right\} + q\frac{d^2}{dq^2}\left\{\frac{1}{1-q}\right\}\right] \\
&= (1-q)\left[\frac{1}{(1-q)^2} + q\frac{d}{dq}\left\{\frac{1}{(1-q)^2}\right\}\right] \\
&= \frac{1}{1-q} + q(1-q)\frac{2(1-q)}{(1-q)^4} \\
&= \frac{1}{1-q} + \frac{2q}{(1-q)^2} \\
&= \frac{1}{p} + \frac{2(1-p)}{p^2} \\
&= \frac{2-p}{p^2}
\end{aligned}
$$

Next, we can easily obtain the variance

$$
\mathrm{Var}\left[X\right] = \mathbb{E}\left[X^2\right] - \left(\mathbb{E}\left[X\right]\right)^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}
$$

(c) Uniform distribution

$$
\begin{aligned}
\mathbb{E}\left[X\right] &= \int_a^b x\frac{1}{b-a}dx \\
&= \left.\frac{x^2}{2(b-a)}\right|_a^b \\
&= \frac{b^2-a^2}{2(b-a)} \\
&= \frac{(b-a)(b+a)}{2(b-a)} \\
&= \frac{(b+a)}{2}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[X^2\right] &= \int_a^b \frac{x^2}{b-a}dx \\
&= \left.\frac{x^3}{3(b-a)}\right|_a^b \\
&= \frac{b^3-a^3}{3(b-a)} \\
&= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} \\
&= \frac{b^2+ab+a^2}{3}
\end{aligned}
$$

Therefore, the variance of X is given by,

$$
\begin{aligned}
\operatorname{Var}[X] &= \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2 \\
&= \frac{b^2+ab+a^2}{3} - \frac{(b+a)^2}{4} \\
&= \frac{(b-a)^2}{12}
\end{aligned}
$$

(d) Exponential distribution:

$$
\begin{aligned}
\mathbb{E}[X] &= \int_0^\infty x\lambda e^{-\lambda x}dx \\
&= -xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x}dx \\
&= \left.-\frac{1}{\lambda}e^{-\lambda x}\right|_0^\infty \\
&= \frac{1}{\lambda}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[X^2\right] &= \int_0^\infty x^2\lambda e^{-\lambda x}dx \\
&= -x^2e^{-\lambda x}\Big|_0^\infty + \int_0^\infty 2xe^{-\lambda x}dx \\
&= \left.-\frac{2x}{\lambda}e^{-\lambda x}\right|_0^\infty + \int_0^\infty \frac{2}{\lambda}e^{-\lambda x}dx \\
&= \left.-\frac{2}{\lambda^2}e^{-\lambda x}\right|_0^\infty \\
&= \frac{2}{\lambda^2}
\end{aligned}
$$

Thus, the variance of X is given by

$$
\operatorname{Var}[X] = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}
$$

5. Assume that X and Y are continuous random variable and let $f_{X|Y}(x|y)$ be the conditional density of

X given Y. Then,

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}\left[X|Y\right]\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx f_Y(y)dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X|Y}(x|y) f_Y(y)dxdy \qquad \text{Note: } f_Y(y) \text{ is independent of } x \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x,y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x,y)dydx \\
&= \int_{-\infty}^{\infty} x f_X(x)dx \\
&= \mathbb{E}\left[X\right]
\end{aligned}$$

A similar proof also holds for the discrete random variable case, but rather than integrating we use a summation.