Homework 2

1. The number of messages successfully reaching a receiver at time $k = 0, 1, 2, \cdots$ is denoted by S_k, and it is modelled as na iid sequence with a distribution specified by $S_k = n$ with probability $1/N$, where $n = 1, 2, \cdots, N$ for some given integer N. The cumulative number of successfully received messages is described by the random sequence

$$X_{k+1} = X_k + S_k, \qquad X_0 = 0$$

   (a) Does $\{X_k\}$ possess the Markov property?

   (b) Calculate the *mean* $m_k = \mathbb{E}[X_k]$ and variance $\sigma_k^2 = \mathbb{E}\left[(X_k - m_k)^2\right]$ of this random sequence.

2. Consider a simple queueing system where customers arrive according to a Poisson process of rate λ. Assume that service times are exponentially distributed with rate μ. Let X denote the number of customers that arrive during a service time. Determine $\Pr\{X = k\}$, $k = 0, 1, 2...$

   Note: The following integral may be of help but it is not necessarily needed to obtain the result

$$\int_0^\infty x^{k-1} e^{-x} dx = \Gamma(k), \qquad k = 1, 2, ...$$

   where the gamma function $\Gamma(k)$ satisfies $\Gamma(k+1) = k!$

3. Given a Poisson process, suppose that exactly n events are observed in the interval $(0, t)$. Consider a sequence of time instants such that $0 < s_1 < t_1 < s_2 < t_2 < \cdots < s_n < t_n < t$. Let $\tau_i = t_i - s_i$, $i = 1, \cdots, n$. Show that the probability that exactly one event occurs in each of the intervals $(s_i, t_i]$, $i = 1, \cdots, n$, given that exactly n events have occurred in $(0, t)$ is

$$n! \prod_{i=1}^n \frac{\tau_i}{t}$$

4. Given the Poisson distribution

$$\Pr\{X(t) = n\} = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad t \geq 0, n = 0, 1, 2, \cdots$$

   set $t = 1$ to obtain the probability distribution that characterizes the random variable X

$$\Pr\{X = n\} = \frac{\lambda^n e^{-\lambda}}{n!}, \qquad n = 0, 1, 2, \cdots$$

   Show that the discrete random variable

$$Y = \sum_{i=1}^N X_i,$$

   where every X_i, $i = 1, \cdots, N$ has the above Poisson distribution with parameter λ and all X_i are mutually independent, also has a Poisson distribution with parameter $N\lambda$.

5. (*) Suppose that $\{X_i\}$ is a sequence of iid random variables according to the exponential distribution with rate λ. Let $Y_n = \sum_{i=1}^n X_i$. Find the distribution of Y_n.

(*) Optional