Homework 2 Solution

1. (a) Clearly, X_k possesses the Markov property since X_{k+1} depends only on the current state X_k and not on any part of previous information.

   (b) Since $S_k = n$, $n = 1, \cdots, N$ with equal probability,

$$\mathbb{E}[S_k] = \sum_{i=1}^{N} \frac{i}{N} = \frac{(N+1)N}{2N} = \frac{N+1}{2}$$

$$\begin{aligned} \mathrm{Var}[S_k] &= \mathbb{E}\left[(X^2)\right] - \frac{(N+1)^2}{4} \\ &= \frac{N(N+1)(2N+1)}{6N} - \frac{(N+1)^2)}{4} \\ &= \frac{N^2-1}{12}. \end{aligned}$$

$$\begin{aligned} m_k = \mathbb{E}[X_k] &= \mathbb{E}[X_{k-1} + S_k] \\ &= \mathbb{E}\left[\sum_{i=1}^{k} S_i\right] \\ &= \sum_{i=1}^{k} \mathbb{E}[S_i] \\ &= \frac{k(N+1)}{2} \end{aligned}$$

$$\begin{aligned} \sigma_k^2 &= \mathbb{E}\left[(X_k - m_k)^2\right] \\ &= \mathbb{E}\left[\left(\sum_{i=1}^{k} S_i - \frac{k(N+1)}{2}\right)^2\right] \\ &= \mathbb{E}\left[\left(\sum_{i=1}^{k}\left(S_i - \frac{(N+1)}{2}\right)\right)^2\right] \\ &= \mathbb{E}\left[\sum_{i=1}^{k}\sum_{j=1}^{k}\left(S_i - \frac{(N+1)}{2}\right)\left(S_j - \frac{(N+1)}{2}\right)\right] \\ &= \mathbb{E}\left[\sum_{i=1}^{k}\left(S_i - \frac{(N+1)}{2}\right)^2 + \sum_{i=1}^{k}\sum_{j=1, j\neq i}^{k}\left(S_i - \frac{(N+1)}{2}\right)\left(S_j - \frac{(N+1)}{2}\right)\right] \\ &= \sum_{i=1}^{k}\mathbb{E}\left[\left(S_i - \frac{(N+1)}{2}\right)^2\right] + \sum_{i=1}^{k}\sum_{j=1, j\neq i}^{k}\mathbb{E}\left[\left(S_i - \frac{(N+1)}{2}\right)\left(S_j - \frac{(N+1)}{2}\right)\right] \\ &= \sum_{i=1}^{k}\mathrm{Var}[S_i] = \frac{k(N^2-1)}{12} \end{aligned}$$

2. Let us denote the service time by S, i.e., S is an exponential random variable with density $f_S(t) = \mu e^{-\mu t}$. Therefore, we can write that

$$\begin{aligned}\Pr\{X = n\} &= \mathbb{E}\left[\Pr\{X = n|S = t\}\right] \\ &= \int_0^\infty \Pr\{X = n|S = t\}\mu e^{-\mu t} dt \qquad (1)\end{aligned}$$

From the Poisson distribution we know that

$$\Pr\{X = n|S = t\} = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

Substituting in (1) we get

$$\Pr\{X = n\} = \int_0^\infty \frac{(\lambda t)^n}{n!} \mu e^{-(\lambda+\mu)t} dt$$

Let $\tau = (\lambda + \mu)t$, therefore $d\tau = (\lambda + \mu)dt$. Using this substitution we get

$$\begin{aligned}\Pr\{X = n\} &= \int_0^\infty \frac{\lambda^n \tau^n}{n!(\lambda + \mu)^n)} \mu e^{-\tau} \frac{d\tau}{\lambda + \mu} \\ &= \frac{\mu \lambda^n}{n!(\lambda + \mu)^{n+1}} \int_0^\infty \tau^n e^{-\tau} d\tau \\ &= \frac{\mu \lambda^n}{n!(\lambda + \mu)^{n+1}} \Gamma(n + 1) \\ &= \frac{\mu \lambda^n}{(\lambda + \mu)^{n+1}}, \quad n = 0, 1, \cdots\end{aligned}$$

3. Let $\mathcal{A}$ denote the event that there is one event in each of the intervals (s_i, t_i), $i = 1, \cdots, n$ and denote the number of events in the interval $(0, t)$ by $N(t)$. What we are after is $\Pr\{\mathcal{A}|N(t) = n\}$. Using the definition of conditional probability

$$\Pr\{\mathcal{A}|N(t) = n\} = \frac{\Pr\{\mathcal{A}, N(t) = n\}}{\Pr\{N(t) = n\}} \qquad (2)$$

Since, this is a Poisson process,

$$\Pr\{N(t) = n\} = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

Next define $N(t_1, t_2) = N(t_2) - N(t_1)$, i.e., it is the number of events in the interval (t_1, t_2), $t_2 \geq t_1$. Therefore,

$$\begin{aligned}\Pr\{\mathcal{A}, N(t) = n\} = \ &\Pr\{N(s_1) = 0, N(s_1, t_1) = 1, \cdots, N(t_{i-1}, s_i) = 0, N(s_i, t_i) = 1, \cdots, \\ &N(t_{n-1}, s_n) = 0, N(s_n, t_n) = 1\}.\end{aligned}$$

Next, using the stationary independent increments assumption of the Poisson process, we get

$$\begin{aligned}\Pr\{\mathcal{A}, N(t) = n\} &= \Pr\{N(s_1) = 0\} \Pr\{N(s_1, t_1) = 1\} \cdots \Pr\{N(t_{i-1}, s_i) = 0\} \Pr\{N(s_i, t_i) = 1\} \cdots \\ &\qquad \Pr\{N(t_{n-1}, s_n) = 0\} \Pr\{N(s_n, t_n) = 1\}. \\ &= \Pr\{N(s_1) = 0\} \Pr\{N(t_1 - s_1) = 1\} \cdots \Pr\{N(s_i - t_{i-1}) = 0\} \Pr\{N(t_i - s_i) = 1\} \cdots \\ &\qquad \Pr\{N(s_n - t_{n-1}) = 0\} \Pr\{N(t_n - s_n) = 1\}. \\ &= \prod_{i=1}^n \Pr\{N(s_i - t_{i-1}) = 0\} \prod_{i=1}^n \Pr\{N(t_i - s_i) = 1\} \\ &= \prod_{i=1}^n \frac{(\lambda(s_i - t_{i-1}))^0\}}{0!} e^{-\lambda(s_i - t_{i-1})} \prod_{i=1}^n \frac{(\lambda \tau_i)^1}{1!} e^{-\lambda \tau_i} \\ &= e^{-\lambda \sum_{i=1}^n (s_i - t_{i-1} + \tau_i)} \lambda^n \prod_{i=1}^n \tau_i \\ &= e^{-\lambda t} \lambda^n \prod_{i=1}^n \tau_i\end{aligned}$$

Substituting this in (2) we get

$$\Pr\{\mathcal{A}|N(t)=n\} = \frac{e^{-\lambda t}\lambda^n \prod_{i=1}^n \tau_i}{\frac{(\lambda t)^n}{n!}e^{-\lambda t}} = n! \prod_{i=1}^n \frac{\tau_i}{t}$$

4. Assume that $\{X_i(t)\}$, $i = 1, \cdots, N$ is a sequence of mutually independent Poisson processes. From the class notes, we know that the superposition of N Poisson processes is also a Poisson process with rate $N\lambda$. Recall that we showed that the event interarrival times of the resulting process is exponentially distributed with rate $N\lambda$ and note that exponential interarrival times imply a Poisson process. Thus $Y(t)$ is a Poisson process with rate $N\lambda$, therefore

$$\Pr\{Y(t)=n\} = \frac{(N\lambda t)^n}{n!}e^{-N\lambda t}$$

Fix, $t = 1$ to obtain,

$$\Pr\{Y=n\} = \frac{(N\lambda)^n}{n!}e^{-N\lambda}$$

5. The moment generating function of the random variable Y_n is given by

$$M_{Y_n}(v) = \mathbb{E}\left[e^{vY_n}\right] = \int_{-\infty}^{\infty} e^{vy} f_{Y_n}(y)dy$$

Since $Y_n = \sum_{i=1}^n X_i$ where X_i and iid

$$M_{Y_n}(v) = \mathbb{E}\left[e^{v\sum_{i=1}^n X_i}\right] = \mathbb{E}\left[\prod_{i=1}^n e^{vX_i}\right] = \prod_{i=1}^n \mathbb{E}\left[e^{vX_i}\right]$$

where the last equality is due to the independence of X_is. The moment generating function of X_is is easily evaluated

$$\begin{aligned} M_X(v) &= \mathbb{E}\left[e^{vX}\right] = \int_0^\infty e^{vx} f_X(x)dx \\ &= \int_0^\infty e^{vx}\lambda e^{-\lambda x}dx \\ &= -\frac{\lambda}{\lambda - v}e^{-(\lambda - v)x}\Big|_0^\infty \\ &= \frac{\lambda}{\lambda - v} \end{aligned}$$

Therefore, the moment generating function of Y_n is given by

$$M_{Y_n}(v) = \left(\frac{\lambda}{\lambda - v}\right)^n.$$

Note that the Laplace transform of the pdf of Y_n is obtained by simply substituting $s = -v$ (where s in a complex number). Therefore,

$$M_{Y_n}(s) = \left(\frac{\lambda}{\lambda + s}\right)^n.$$

From any Laplace transform table one can obtain the distribution of Y_n as the inverse of the above transform

$$f_{Y_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!}e^{-\lambda t} = \frac{\lambda(\lambda t)^{n-1}}{(n-1)!}e^{-\lambda t}, \quad t \geq 0.$$

This corresponds to the well known Erlang distribution of order n. Note that for $n = 1$ we get the exponential distribution.