Homework 3

1. The following game is played with a regular 52-card deck. A card is drawn with a player trying to guess its color (red or black). If the player guesses right, he gets 1 dollar, otherwise he gets nothing. After a card is drawn it is set aside (it is not placed back in the deck). Suppose the player decides to adopt the simple policy "guess red all the time", and let X_k denote his total earnings after k cards are drawn, with $X_0 = 0$.

   (a) Is $\{X_k\}$ a Markov chain? If so, is it a homogeneous Markov chain?

   (b) Compute $P[X_7 = 6 \mid X_6 = 5]$.

   (c) Compute $P[X_9 = 4 \mid X_8 = 2]$.

   (d) Derive an expression for $P[X_{k+1} = j \mid X_k = i]$ for all $i, j = 0, 1, 2, \ldots$, and $k = 0, 1, \ldots, 51$.

2. The following is used to model the genetic evolution of various types of populations. The term "cell" is used to denote a unit in that population. Let X_k be the total number of cells in the kth generation, $k = 1, 2, \ldots$ The ith cell in a particular generation acts independently of all other cells and it either dies or it splits up into a random number of offspring. Let Y_i be an integer-valued non-negative random variable, so that $Y_i = 0$ indicates that the ith cell dies without reproducing, and $Y_i > 0$ indicates the resulting number of cells after reproduction. If $X_k = n$ and $Y_i = 0$ for all cells $i = 1, \ldots, n$ in the kth generation, we get $X_{k+1} = 0$, and we say that the population becomes extinct.

   (a) Is $\{X_k\}$ a Markov chain? If so, is it a homogeneous Markov chain?

   (b) Suppose $X_0 = 1$ and the random variable Y_i can only take three values, 0, 1, or 2, each with probability 1/3. Draw a state transition diagram for the chain and calculate the transition probabilities $P[X_{k+1} = j \mid X_k = i]$ for $i = 0, 1, 2, 3$.

   (c) Compute the probability that the population becomes extinct after four generations.

   (d) Compute the probability that the population remains in existence for more than three generations.

3. The weather in some area is classified as either "sunny", "cloudy", or "rainy" on any given day. Let X_k denote the weather state on the kth day, $k = 1, 2, \ldots$, and map these three states into the numbers 0, 1, 2 respectively. Suppose transition probabilities from one day to the next are summarized through the matrix:

$$\mathbf{P} = \begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.5 & 0.3 & 0.2 \\ 0.1 & 0.5 & 0.4 \end{bmatrix}$$

   (a) Draw a state transition diagram for this chain.

   (b) Assuming the weather is cloudy today, predict the weather for the next two days.

   (c) Find the stationary state probabilities (if they exist) for this chain. If they do not exist, explain why this is so.

   (d) If today is a sunny day, find the average number of days we have to wait at steady state until the next sunny day.

4. A random number of messages A_k is placed in a transmitter at times $k = 0, 1, \ldots$ Each message present at time k acts independently of past arrivals and of other messages, and it is successfully transmitted with probability p. Thus, if X_k denotes the number of messages present at k, and $D_k \leq X_k$ denotes the number of successful transmissions at k, we can write

$$X_{k+1} = X_k - D_k + A_{k+1}$$

Assume that A_k has a Poisson distribution $P[A_k = n] = \lambda^n e^{-\lambda}/n!, n = 0, 1, \ldots$, independent of k.

(a) If the system is at state $X_k = i$, find $P[D_k = m \mid X_k = i], m = 0, 1, \ldots$

(b) Determine the transition probabilities $P[X_{k+1} = j \mid X_k = i], i, j = 0, 1, \ldots$

(c) If $X_0 = 0$, show that X_1 has a Poisson distribution with parameter λ, and that $(X_1 - D_1)$ has a Poisson distribution with parameter $\lambda(1-p)$. Hence, determine the distribution of X_2.

(d) If X_0 has a Poisson distribution with parameter ν, find the value of ν such that there exists a stationary probability distribution for the Markov chain $\{X_k\}$.

5. Consider an $M/M/1$ queueing system where customers are "discouraged" from entering the queue in the following fashion: While the queue length is less than or equal to K for some $K > 0$, the arrival rate is fixed at λ; when the queue length is greater than K, the arrival rate becomes $\lambda_n = \lambda/n, n \geq K+1$. The service rate is fixed at $\mu > \lambda$.

(a) Find the stationary probability distribution of the queue length.

(b) Determine the average queue length and system time at steady state.

(c) How far can you reduce the service rate before the average queue length becomes infinite?

6. The time to process transactions in a computer system is exponentially distributed with mean 6 sec. There are two different sources from which transactions originate, both modeled as Poisson processes which are independent of each other and of the state of the system. The first source generates transactions at a rate of 4 per minute, and the second source at a rate of 3 per minute. Up to K transactions (waiting to be processed or in process) can be queued in this system. When a transaction finds no queueing space (i.e., it is *blocked*) it is rejected and lost. Determine the minimum value of K such that the blocking probability for source 1 does not exceed 10%, and that of source 2 does not exceed 2%. Once the value of K is determined, calculate the average response time (system time) for tasks that actually get processed.

7. (*) A manufacturing process involves operations performed on five machines $M_1, \ldots, M_5$. The processing times at the five machines are modeled through exponential distributions with rates $\mu_1 = 2$ parts/min, $\mu_2 = 1.75$ parts/min, $\mu_3 = 2.2$ parts/min, $\mu_4 = 1$ part/min, and $\mu_5 = 0.5$ parts/min. All parts first arrive at M_1 according to a Poisson process with rate 1 part/min. After being processed at M_1, a fraction $p_{14} = 0.2$ of all parts is sent to machine M_4, while the rest proceed to M_2. All parts processed at M_4 are sent back to M_1 for one more cycle at that machine. Parts processed at M_2 are all routed to M_3. After being processed at M_3, a fraction $p_{34} = 0.2$ of all parts is sent back to M_4, another fraction $p_{35} = 0.2$ is sent to machine M_5, and all other parts leave the system. Finally, all parts processed at M_5 are sent back to M_2 for more processing.

(a) Obtain expressions for the utilizations $\rho_1, \ldots, \rho_5$ at all five machines in terms of the parameters $\mu_1, \ldots, \mu_5$, λ, and p_{14}, p_{34}, p_{35}.

(b) Calculate the throughput of this system (i.e., the rate of finished part departures from M_3).

(c) Calculate the average system time of parts in the system.

(d) Find the probability that the queue length at machine M_2 is greater than 3 parts.

8. Consider a simple routing problem where customers arrive according to a Poisson process with rate λ. Every arriving customer is routed to queue 1 with probability p and to queue 2 with probability $1-p$. The problem is to determine the value of p such that the average system time of customers in this system is minimized (both queues have infinite capacity).

(a) Assuming service times at both servers are exponentially distributed with parameters μ_1 and μ_2, find an expression for the value of p that solves this optimization problem. For $\lambda = 2, \mu_1 = 1, \mu_2 = 1.5$, calculate this value of p.

(b) Repeat part (a), except now let server 2 be deterministic, and assume that the service time is fixed at $1/\mu_2$.

(c) For the numerical values given, calculate the average system time under the optimal p in part (a) and in part (b). Compare your results and comment on whether their relative value is as you expected.

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