Markov Chains

Summary

- Markov Chains
- Discrete Time Markov Chains
  - Homogeneous and non-homogeneous Markov chains
  - Transient and steady state Markov chains
- Continuous Time Markov Chains
  - Homogeneous and non-homogeneous Markov chains
  - Transient and steady state Markov chains

Markov Processes

- Recall the definition of a Markov Process
  - The future a process does not depend on its past, only on its present

$$\Pr\{X(t_{k+1}) \le x_{k+1} \mid X(t_k) = x_k, \ldots, X(t_0) = x_0\}$$
$$= \Pr\{X(t_{k+1}) \le x_{k+1} \mid X(t_k) = x_k\}$$

- Since we are dealing with "chains", $X(t)$ can take discrete values from a finite or a countable infinite set.
- For a discrete-time Markov chain, the notation is also simplified to

$$\Pr\{X_{k+1} = x_{k+1} \mid X_k = x_k, \ldots, X_0 = x_0\} = \Pr\{X_{k+1} = x_{k+1} \mid X_k = x_k\}$$

- Where X_k is the value of the state at the kth step

Chapman-Kolmogorov Equations

- Define the one-step transition probabilities

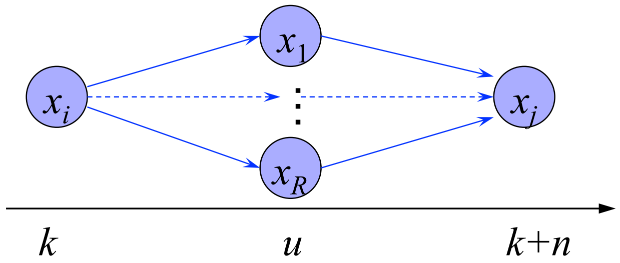
$$p_{ij}(k) = \Pr\{X_{k+1} = j \mid X_k = i\}$$

- Clearly, for all i, k, and all feasible transitions from state i

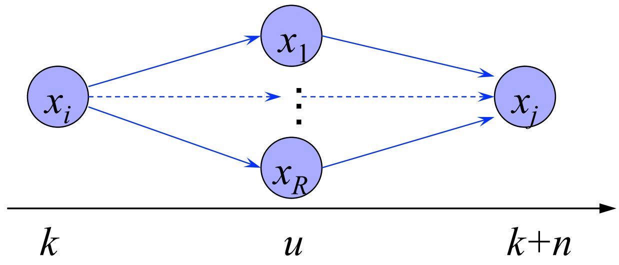
$$\sum_{j \in \Gamma(i)} p_{ij}(k) = 1$$

- Define the n-step transition probabilities

$$p_{ij}(k, k+n) = \Pr\{X_{k+n} = j \mid X_k = i\}$$



Chapman-Kolmogorov Equations



- Using total probability

$$p_{ij}(k,k+n)=\sum_{r=1}^{R}\Pr\{X_{k+n}=j\mid X_u=r,X_k=i\}\Pr\{X_u=r\mid X_k=i\}$$

- Using the memoryless property of Marckov chains

$$\Pr\{X_{k+n}=j\mid X_u=r,X_k=i\}=\Pr\{X_{k+n}=j\mid X_u=r\}$$

- Therefore, we obtain the Chapman-Kolmogorov Equation

$$p_{ij}(k,k+n)=\sum_{r=1}^{R}p_{ir}(k,u)\,p_{rj}(u,k+n),\qquad k\le u\le k+n$$

Matrix Form

- Define the matrix

$$\mathbf{H}(k,k+n)=\left[p_{ij}(k,k+n)\right]$$

- We can re-write the Chapman-Kolmogorov Equation

$$\mathbf{H}(k,k+n)=\mathbf{H}(k,u)\mathbf{H}(u,k+n)$$

- Choose, $u = k+n-1$, then

$$\begin{aligned}\mathbf{H}(k,k+n)&=\mathbf{H}(k,k+n-1)\mathbf{H}(k+n-1,k+n)\\&=\mathbf{H}(k,k+n-1)\mathbf{P}(k+n-1)\end{aligned}$$

Forward Chapman-Kolmogorov

One step transition probability

Matrix Form

- Choose, $u = k+1$, then

$$\mathbf{H}(k,k+n) = \mathbf{H}(k,k+1)\mathbf{H}(k+1,k+n)$$
$$= \mathbf{P}(k)\mathbf{H}(k+1,k+n)$$

Backward Chapman-Kolmogorov

One step transition probability

Homogeneous Markov Chains

- The one-step transition probabilities are independent of time k.

$$\mathbf{P}(k) = \mathbf{P} \qquad \text{or} \qquad \left[p_{ij}\right] = \left[\Pr\left\{X_{k+1} = j \mid X_k = i\right\}\right]$$

- Even though the one step transition is independent of k, this does not mean that the joint probability of X_{k+1} and X_k is also independent of k
  - Note that

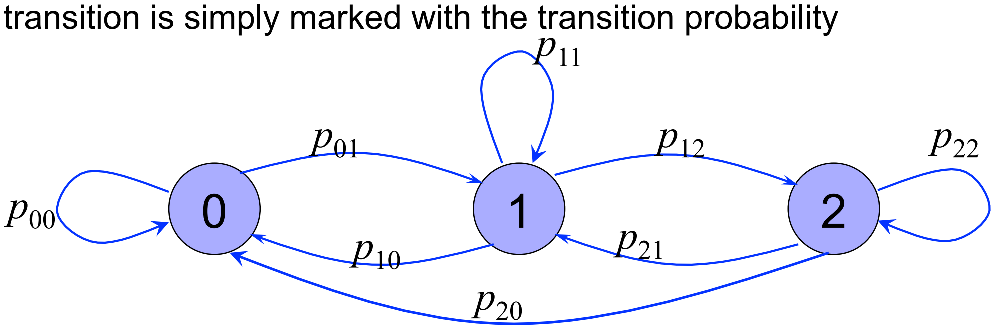
$$\Pr\left\{X_{k+1} = j, X_k = i\right\} = \Pr\left\{X_{k+1} = j \mid X_k = i\right\}\Pr\left\{X_k = i\right\}$$
$$= p_{ij} \Pr\left\{X_k = i\right\}$$

Example

- Consider a two transmitter (Tx) communication system where, time is divided into time slots and that operates as follows
  - At most one packet can arrive during any time slot and this can happen with probability α.
  - Packets are transmitted by whichever transmitter is available, and if both are available then the packet is given to Tx 1.
  - If both transmitters are busy, then the packet is lost
  - When a Tx is busy, it can complete the transmission with probability β during any one time slot.
  - If a packet is submitted during a slot when both transmitters are busy but at least one Tx completes a packet transmission, then the packet is accepted (departures occur before arrivals).
- Describe the Markov Chain that describe this model.

Example: Markov Chain

- For the State Transition Diagram of the Markov Chain, each transition is simply marked with the transition probability



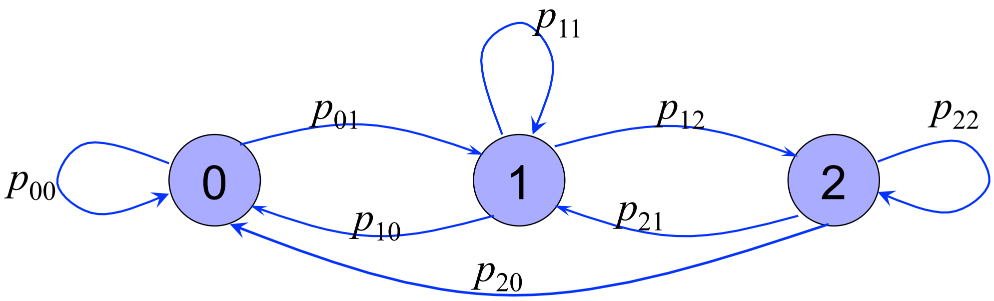
$$p_{00} = (1-\alpha) \qquad p_{01} = \alpha \qquad p_{02} = 0$$

$$p_{10} = \beta(1-\alpha) \qquad p_{11} = (1-\beta)(1-\alpha) + \alpha\beta \qquad p_{12} = \alpha(1-\beta)$$

$$p_{20} = \beta^2(1-\alpha) \qquad p_{21} = \beta^2\alpha + 2\beta(1-\beta)(1-\alpha)$$

$$p_{22} = (1-\beta)^2 + 2\alpha\beta(1-\beta)$$

Example: Markov Chain



- Suppose that $\alpha = 0.5$ and $\beta = 0.7$, then,

$$\mathbf{P} = \left[p_{ij} \right] = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.35 & 0.5 & 0.15 \\ 0.245 & 0.455 & 0.3 \end{bmatrix}$$

State Holding Times

- Suppose that at point k, the Markov Chain has transitioned into state $X_k = i$. An interesting question is how long it will stay at state i.
- Let $V(i)$ be the random variable that represents the number of time slots that $X_k = i$.
- We are interested in the quantity $\Pr\{V(i) = n\}$

$$\begin{aligned} \Pr\{V(i) = n\} &= \Pr\{X_{k+n} \neq i, X_{k+n-1} = i, \ldots, X_{k+1} = i \mid X_k = i\} \\ &= \Pr\{X_{k+n} \neq i \mid X_{k+n-1} = i, \ldots, X_k = i\} \times \\ &\qquad \Pr\{X_{k+n-1} = i, \ldots, X_{k+1} = i \mid X_k = i\} \\ &= \Pr\{X_{k+n} \neq i \mid X_{k+n-1} = i\} \times \\ &\qquad \Pr\{X_{k+n-1} = i \mid X_{k+n-2} \ldots, X_k = i\} \times \\ &\qquad \Pr\{X_{k+n-2} = i, \ldots, X_{k+1} = i \mid X_k = i\} \end{aligned}$$

State Holding Times

$$\Pr\{V(i)=n\}=\Pr\{X_{k+n}\neq i \mid X_{k+n-1}=i\}\times$$
$$\Pr\{X_{k+n-1}=i \mid X_{k+n-2},\ldots,X_k=i\}\times$$
$$\Pr\{X_{k+n-2}=i,\ldots,X_{k+1}=i \mid X_k=i\}$$
$$=(1-p_{ii})\Pr\{X_{k+n-1}=i \mid X_{k+n-2}=i\}\times$$
$$\Pr\{X_{k+n-2}=i \mid X_{k+n-3}=i,\ldots,X_k=i\}$$
$$\Pr\{X_{k+n-3}=i,\ldots,X_{k+1}=i \mid X_k=i\}$$

$$\Pr\{V(i)=n\}=(1-p_{ii})p_{ii}^{n-1}$$

- This is the Geometric Distribution with parameter p_{ii}.
- $V(i)$ has the memoryless property

State Probabilities

- An interesting quantity we are usually interested in is the probability of finding the chain at various states, i.e., we define

$$\pi_i(k)\equiv\Pr\{X_k=i\}$$

- For all possible states, we define the vector

$$\boldsymbol{\pi}(k)=[\pi_0(k),\pi_1(k)\ldots]$$

- Using total probability we can write

$$\pi_i(k)=\sum_j \Pr\{X_k=i \mid X_{k-1}=j\}\Pr\{X_{k-1}=j\}$$
$$=\sum_j p_{ij}(k)\pi_j(k-1)$$

- In vector form, one can write

$$\boldsymbol{\pi}(k)=\boldsymbol{\pi}(k-1)\mathbf{P}(k)$$ Or, if homogeneous Markov Chain $$\boldsymbol{\pi}(k)=\boldsymbol{\pi}(k-1)\mathbf{P}$$

State Probabilities Example

- Suppose that

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.35 & 0.5 & 0.15 \\ 0.245 & 0.455 & 0.3 \end{bmatrix} \quad \text{with} \quad \boldsymbol{\pi}(0) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

- Find $\boldsymbol{\pi}(k)$ for k=1,2,…

$$\boldsymbol{\pi}(1) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.35 & 0.5 & 0.15 \\ 0.245 & 0.455 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0 \end{bmatrix}$$

- *Transient* behavior of the system: `MCTransient.m`
- In general, the transient behavior is obtained by solving the difference equation

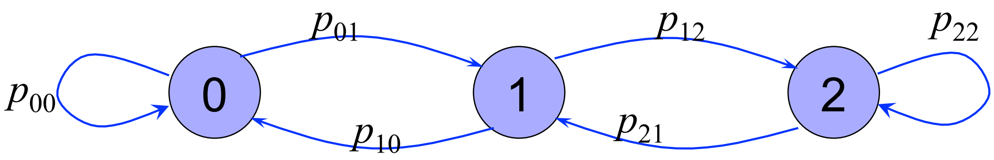
$$\boldsymbol{\pi}(k) = \boldsymbol{\pi}(k-1)\mathbf{P}$$

Classification of States

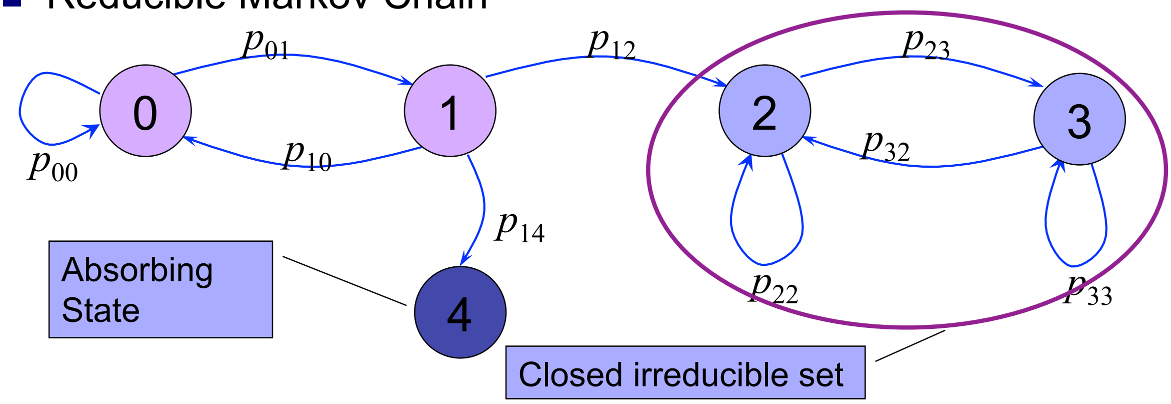
- Definitions
  - State j is **reachable** from state i if the probability to go from i to j in $n>0$ steps is greater than zero (State j is reachable from state i if in the state transition diagram there is a path from i to j).
  - A subset S of the state space X is **closed** if $p_{ij}=0$ for every $i \in S$ and $j \notin S$
  - A state i is said to be **absorbing** if it is a single element closed set.
  - A closed set S of states is **irreducible** if any state $j \in S$ is reachable from every state $i \in S$.
  - A Markov chain is said to be **irreducible** if the state space X is irreducible.

Example

- Irreducible Markov Chain



- Reducible Markov Chain



Transient and Recurrent States

- Hitting Time $T_{ij} = \min\{k > 0 : X_0 = i, X_k = j\}$
- **Recurrence Time** T_{ii} is the first time that the MC returns to state i.
- Let ρ_i be the probability that the state will return back to i given it starts from i. Then,

$$\rho_i = \sum_{k=1}^{\infty} \Pr\{T_{ii} = k\}$$

- The event that the MC will return to state i given it started from i is equivalent to $T_{ii} < \infty$, therefore we can write

$$\rho_i = \sum_{k=1}^{\infty} \Pr\{T_{ii} = k\} = \Pr\{T_{ii} < \infty\}$$

- A state is **recurrent** if $\rho_i = 1$ and **transient** if $\rho_i < 1$

Theorems

- If a Markov Chain has finite state space, then at least one of the states is recurrent.
- If state i is recurrent and state j is reachable from state i then, state j is also recurrent.
- If S is a finite closed irreducible set of states, then every state in S is recurrent.

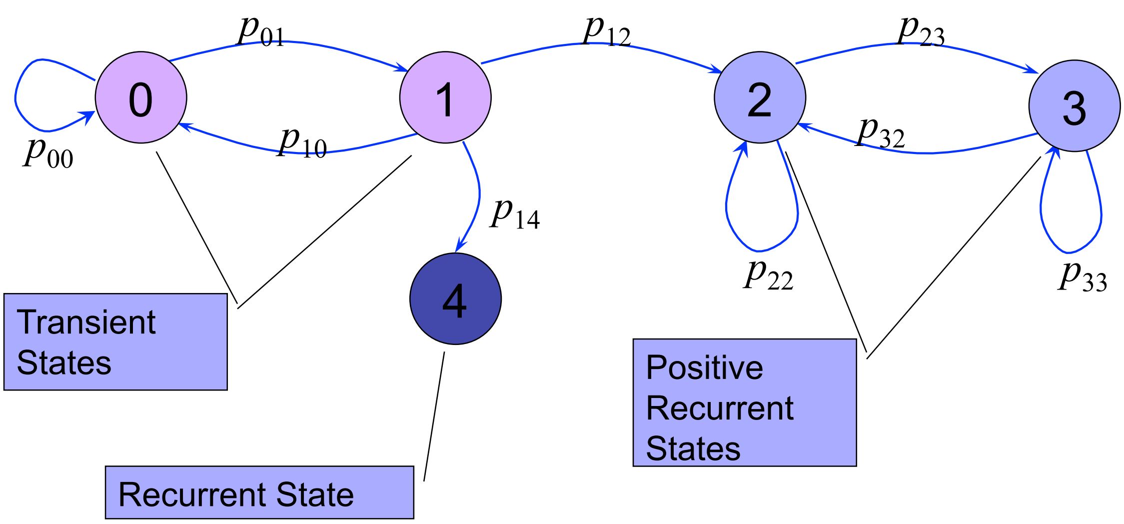
Positive and Null Recurrent States

- Let M_i be the mean recurrence time of state i

$$M_i \equiv E[T_{ii}] = \sum_{k=1}^{\infty} k \Pr\{T_{ii} = k\}$$

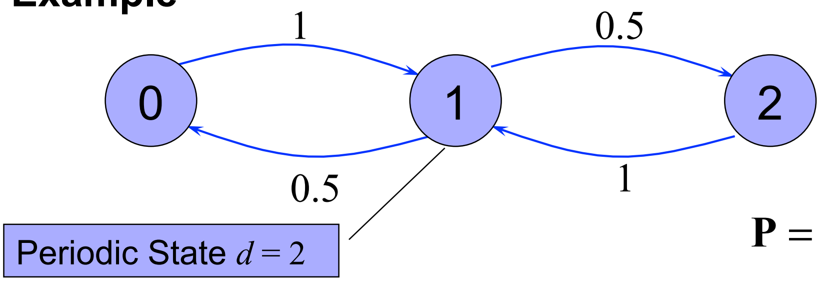
- A state is said to be **positive recurrent** if $M_i < \infty$. If $M_i = \infty$ then the state is said to be **null-recurrent**.
- Theorems
  - If state i is positive recurrent and state j is reachable from state i then, state j is also positive recurrent.
  - If S is a closed irreducible set of states, then every state in S is positive recurrent or, every state in S is null recurrent, or, every state in S is transient.
  - If S is a finite closed irreducible set of states, then every state in S is positive recurrent.

Example



Periodic and Aperiodic States

- Suppose that the structure of the Markov Chain is such that state i is visited after a number of steps that is an integer multiple of an integer $d > 1$. Then the state is called **periodic** with period d.
- If no such integer exists (i.e., $d = 1$) then the state is called **aperiodic**.
- **Example**



$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}$$

Steady State Analysis

- Recall that the probability of finding the MC at state i after the kth step is given by

$$\pi_i(k) \equiv \Pr\{X_k = i\} \qquad \boldsymbol{\pi}(k) = [\pi_0(k), \pi_1(k) \ldots]$$

- An interesting question is what happens in the "long run", i.e., $\pi_i \equiv \lim_{k \to \infty} \pi_i(k)$
- This is referred to as **steady state** or **equilibrium** or **stationary state** probability
- Questions:
  - Do these limits exists?
  - If they exist, do they converge to a legitimate probability distribution, i.e., $\sum \pi_i = 1$
  - How do we evaluate π_j, for all j.

Steady State Analysis

- Recall the recursive probability

$$\boldsymbol{\pi}(k+1) = \boldsymbol{\pi}(k)\mathbf{P}$$

- If steady state exists, then $\boldsymbol{\pi}(k+1) = \boldsymbol{\pi}(k)$, and therefore the steady state probabilities are given by the solution to the equations

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P} \quad \text{and} \quad \sum_i \pi_i = 1$$

- For Irreducible Markov Chains the presence of periodic states prevents the existence of a steady state probability
- Example: `periodic.m`

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix} \qquad \boldsymbol{\pi}(0) = [1 \quad 0 \quad 0]$$

Steady State Analysis

- **THEOREM**: If an irreducible aperiodic Markov chain consists of *positive recurrent* states, a unique stationary state probability vector $\boldsymbol{\pi}$ exists such that $\pi_j > 0$ and

$$\pi_j = \lim_{k \to \infty} \pi_j(k) = \frac{1}{M_j}$$

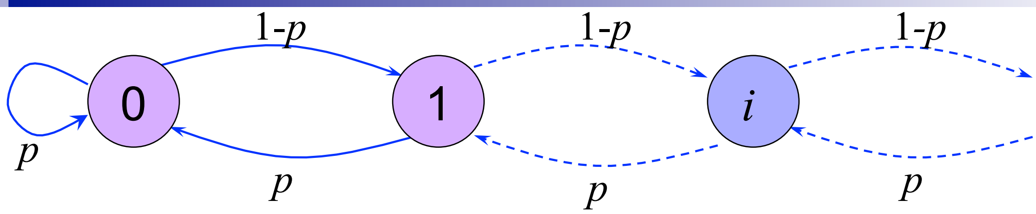
where M_j is the mean recurrence time of state j

- The steady state vector $\boldsymbol{\pi}$ is determined by solving

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P} \quad \text{and} \quad \sum_i \pi_i = 1$$

- Ergodic Markov chain.

Birth-Death Example



$$\mathbf{P} = \begin{bmatrix} p & 1-p & 0 & \cdots \\ p & 0 & 1-p & \cdots \\ 0 & p & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- Thus, to find the steady state vector $\boldsymbol{\pi}$ we need to solve

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P} \quad \text{and} \quad \sum_i \pi_i = 1$$

Birth-Death Example

- In other words

$$\pi_0 = \pi_0 p + \pi_1 p$$
$$\pi_j = \pi_{j-1}(1-p) + \pi_{j+1} p, j = 1, 2, ...$$

- Solving these equations we get

$$\pi_1 = \frac{1-p}{p}\pi_0 \qquad \pi_2 = \left(\frac{1-p}{p}\right)^2 \pi_0$$

- In general $\pi_j = \left(\frac{1-p}{p}\right)^j \pi_0$

- Summing all terms we get

$$\pi_0 \sum_{i=0}^{\infty}\left(\frac{1-p}{p}\right)^i = 1 \Rightarrow \pi_0 = 1 \Big/ \sum_{i=0}^{\infty}\left(\frac{1-p}{p}\right)^i$$

Birth-Death Example

- Therefore, for all states j we get

$$\pi_j = \left(\frac{1-p}{p}\right)^j \Big/ \sum_{i=0}^{\infty}\left(\frac{1-p}{p}\right)^i$$

- If $p<1/2$, then

$$\sum_{i=0}^{\infty}\left(\frac{1-p}{p}\right)^i = \infty \qquad \Rightarrow \pi_j = 0, \quad \text{for all } j$$

All states are *transient*

- If $p>1/2$, then

$$\sum_{i=0}^{\infty}\left(\frac{1-p}{p}\right)^i = \frac{p}{2p-1} > 0 \qquad \Rightarrow \pi_j = \frac{2p-1}{p}\left(\frac{1-p}{p}\right)^j, \quad \text{for all } j$$

All states are *positive recurrent*

Birth-Death Example

- If p=1/2, then

$$\sum_{i=0}^{\infty}\left(\frac{1-p}{p}\right)^{i}=\infty \qquad \Rightarrow \pi_j = 0, \quad \text{for all } j$$

All states are *null recurrent*

Continuous-Time Markov Chains

- In this case, transitions can occur at **any** time
- Recall the Markov (memoryless) property

$$\Pr\left\{X\left(t_{k+1}\right)=x_{k+1} \mid X\left(t_k\right)=x_k,\ldots,X\left(t_0\right)=x_0\right\}$$
$$=\Pr\left\{X\left(t_{k+1}\right)=x_{k+1} \mid X\left(t_k\right)=x_k\right\}$$

where $t_1 < t_2 < \ldots < t_k$

- Recall that the Markov property implies that
  - $X(t_{k+1})$ depends only on $X(t_k)$ (state memory)
  - It does not matter how long the state is at $X(t_k)$ (age memory).
- The transition probabilities now need to be defined for every time instant as $p_{ij}(t)$, i.e., the probability that the MC transitions from state i to j at time t.

Transition Function

- Define the transition function

$$p_{ij}(s,t) \equiv \Pr\{X(t)=j \mid X(s)=i\}, \quad s \le t$$

- The continuous-time analogue of the Chapman-Kolmokorov equation is

$$p_{ij}(s,t) \equiv$$

$$\sum_r \Pr\{X(t)=j \mid X(u)=r, X(s)=i\}\Pr\{X(u)=r \mid X(s)=i\}$$

- Using the memoryless property

$$p_{ij}(s,t) \equiv \sum_r \Pr\{X(t)=j \mid X(u)=r\}\Pr\{X(u)=r \mid X(s)=i\}$$

- Define $\mathbf{H}(s,t)=[p_{ij}(s,t)]$, $i,j=1,2,\ldots$ then

$$\mathbf{H}(s,t)=\mathbf{H}(s,u)\mathbf{H}(u,t), \quad s \le u \le t$$

  - Note that $\mathbf{H}(s, s)= \mathbf{I}$

Transition Rate Matrix

- Consider the Chapman-Kolmogorov for $s \le t \le t+\Delta t$

$$\mathbf{H}(s,t+\Delta t)=\mathbf{H}(s,t)\mathbf{H}(t,t+\Delta t)$$

- Subtracting $\mathbf{H}(s,t)$ from both sides and dividing by Δt

$$\frac{\mathbf{H}(s,t+\Delta t)-\mathbf{H}(s,t)}{\Delta t}=\frac{\mathbf{H}(s,t)\left(\mathbf{H}(t,t+\Delta t)-\mathbf{I}\right)}{\Delta t}$$

- Taking the limit as $\Delta t \rightarrow 0$

$$\frac{\partial \mathbf{H}(s,t)}{\partial t}=\mathbf{H}(s,t)\mathbf{Q}(t)$$

where the **transition rate** matrix $\mathbf{Q}(t)$ is given by

$$\mathbf{Q}(t)=\lim_{\Delta t \to 0}\frac{\mathbf{H}(t,t+\Delta t)-\mathbf{I}}{\Delta t}$$

Homogeneous Case

- In the homogeneous case, the transition functions do not depend on s and t, but only on the difference t-s thus

$$p_{ij}(s,t) = p_{ij}(t-s)$$

- It follows that

$$\mathbf{H}(s,t) = \mathbf{H}(t-s) \equiv \mathbf{P}(\tau)$$

and the **transition rate** matrix

$$\mathbf{Q}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{H}(t,t+\Delta t) - \mathbf{I}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{H}(\Delta t) - \mathbf{I}}{\Delta t} = \mathbf{Q}, \quad \text{constant}$$

- Thus

$$\frac{\partial \mathbf{P}(t)}{\partial t} = \mathbf{P}(t)\mathbf{Q} \;\; \text{with}\; p_{ij}(0) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad \Rightarrow \mathbf{P}(t) = e^{\mathbf{Q}t}$$

State Holding Time

- The time the MC will spend at each state is a random variable with distribution

$$G_i(t) = 1 - e^{-\Lambda_i}$$

where

$$\Lambda_i = \sum_{j \in \Gamma(i)} \lambda_j$$

- Explain why…

Transition Rate Matrix $\mathbf{Q}$.

- Recall that

$$\frac{\partial \mathbf{P}(t)}{\partial t} = \mathbf{P}(t)\mathbf{Q} \Rightarrow \frac{\partial p_{ij}(t)}{\partial t} = \sum_r p_{ir}(t) q_{rj}$$

- First consider the q_{ij}, $i \neq j$, thus the above equation can be written as

$$\frac{\partial p_{ij}(t)}{\partial t} = p_{ii}(t) q_{ij} + \sum_{r \neq i} p_{ir}(t) q_{rj}$$

- Evaluating this at $t = 0$, we get that

$$\left.\frac{\partial p_{ij}(t)}{\partial t}\right|_{t=0} = q_{ij}$$

$p_{ij}(0)= 0$ for all $i \neq j$

- The event that will take the state from i to j has exponential residual lifetime with rate λ_{ij}, therefore, given that in the interval $(t,t+\tau)$ one event has occurred, the probability that this transition will occur is given by $G_{ij}(\tau)=1\text{-}\exp\{-\lambda_{ij}\tau\}$.

Transition Rate Matrix $\mathbf{Q}$.

- Since $G_{ij}(\tau)=1\text{-}\exp\{-\lambda_{ij}\tau\}$.

$$\left.\frac{\partial p_{ij}(\tau)}{\partial \tau}\right|_{\tau=0} = q_{ij} = \left.\lambda_{ij} e^{\lambda_{ij}\tau}\right|_{\tau=0} = \lambda_{ij}$$

- In other words q_{ij} is the rate of the Poisson process that activates the event that makes the transition from i to j.
- Next, consider the q_{jj}, thus

$$\frac{\partial p_{ij}(t)}{\partial t} = p_{ij}(t) q_{jj} + \sum_{r \neq j} p_{ir}(t) q_{rj}$$

- Evaluating this at $t = 0$, we get that

$$\left.\frac{\partial p_{jj}(t)}{\partial t}\right|_{t=0} = q_{jj} \Leftrightarrow \left.\frac{\partial}{\partial t}\left[1 - p_{ij}(t)\right]\right|_{t=0} = -q_{jj}$$

Probability that chain *leaves* state j

Transition Rate Matrix **Q**.

- The event that the MC will transition out of state i has exponential residual lifetime with rate $\Lambda(i)$, therefore, the probability that an event will occur in the interval $(t,t+\tau)$ given by $G_i(\tau)=1\text{-}\exp\{-\Lambda(i)\tau\}$.

$$-q_{jj} = \Lambda(i)e^{-\Lambda(i)\tau}\Big|_{\tau=0} = \Lambda(i)$$

- Note that for each row i, the sum

$$\sum_j q_{ij} = 0$$

Transition Probabilities **P**.

- Suppose that state transitions occur at random points in time $T_1 < T_2 <\ldots< T_k <\ldots$
- Let X_k be the state after the transition at T_k
- Define

$$P_{ij} = \Pr\{X_{k+1} = j \mid X_k = i\}$$

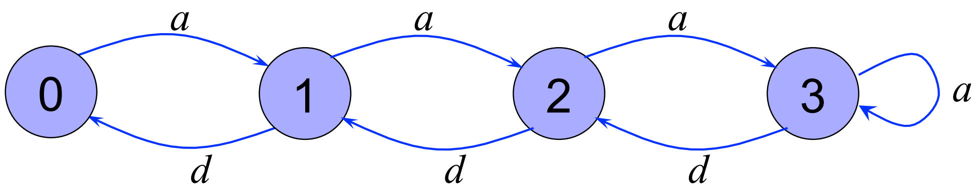
- Recall that in the case of the superposition of two or more Poisson processes, the probability that the next event is from process j is given by λ_j/Λ.
- In this case, we have

$$P_{ij} = \frac{q_{ij}}{-q_{ii}}, i \neq j \qquad \text{and} \qquad P_{ii} = 0$$

Example

- Assume a transmitter where packets arrive according to a Poisson process with rate λ.
- Each packet is processed using a First In First Out (FIFO) policy.
- The transmission time of each packet is exponential with rate μ.
- The transmitter has buffer to store up to two packets that wait to be transmitted.
- Packets that find the buffer full are lost.
- Draw the state transition diagram.
- Find the Rate Transition Matrix $\mathbf{Q}$.
- Find the State Transition Matrix P

Example



- The rate transition matrix is given by

$$\mathbf{Q} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 \\ 0 & \mu & -(\lambda+\mu) & \lambda \\ 0 & 0 & \mu & -\mu \end{bmatrix}$$

- The state transition matrix is given by

$$P = \frac{1}{(\lambda+\mu)} \begin{bmatrix} 0 & (\lambda+\mu) & 0 & 0 \\ \mu & 0 & \lambda & 0 \\ 0 & \mu & 0 & \lambda \\ 0 & 0 & (\lambda+\mu) & 0 \end{bmatrix}$$

State Probabilities and Transient Analysis

- Similar to the discrete-time case, we define

$$\pi_j(t) \equiv \Pr\{X(t) = j\}$$

- In vector form

$$\boldsymbol{\pi}(t) = [\pi_1(t), \pi_2(t), \ldots]$$

- With initial probabilities

$$\boldsymbol{\pi}(0) = [\pi_1(0), \pi_2(0), \ldots]$$

- Using our previous notation (for homogeneous MC)

$$\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\mathbf{P}(t) = \boldsymbol{\pi}(0)e^{\mathbf{Q}t}$$

Obtaining a general solution is not easy!

- Differentiating with respect to t gives us more “inside”

$$\frac{d\boldsymbol{\pi}(t)}{dt} = \boldsymbol{\pi}(t)\mathbf{Q} \Leftrightarrow \frac{d\pi_j(t)}{dt} = q_{jj}\pi_j(t) + \sum_{i \neq j} q_{ij}\pi_i(t)$$

“Probability Fluid” view

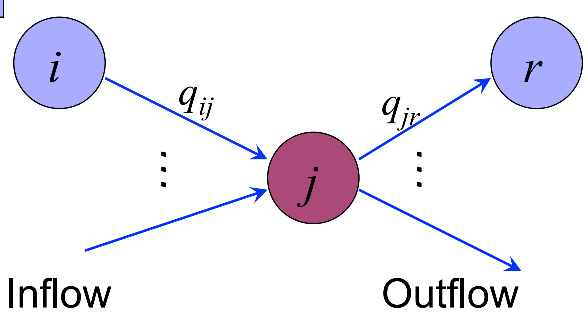
- We view $\pi_j(t)$ as the level of a “probability fluid” that is stored at each node j (0=empty, 1=full).

$$\frac{d\pi_j(t)}{dt} = q_{jj}\pi_j(t) + \sum_{i \neq j} q_{ij}\pi_i(t)$$

Change in the probability fluid

outflow

inflow



$$-q_{jj} = \sum_{r \neq j} q_{jr}$$

Steady State Analysis

- Often we are interested in the "long-run" probabilistic behavior of the Markov chain, i.e.,

$$\pi_j = \lim_{t \to \infty} \pi_j(t)$$

- These are referred to as *steady state probabilities* or *equilibrium state probabilities* or *stationary state probabilities*
- As with the discrete-time case, we need to address the following questions
  - Under what conditions do the limits exist?
  - If they exist, do they form legitimate probabilities?
  - How can we evaluate these limits?

Steady State Analysis

- **Theorem**: In an irreducible continuous-time Markov Chain consisting of positive recurrent states, a unique stationary state probability vector $\boldsymbol{\pi}$ with

$$\pi_j = \lim_{t \to \infty} \pi_j(t)$$

- These vectors are independent of the initial state probability and can be obtained by solving

$$\boldsymbol{\pi}\mathbf{Q} = \mathbf{0} \qquad \text{and} \qquad \sum_j \pi_j = 1$$

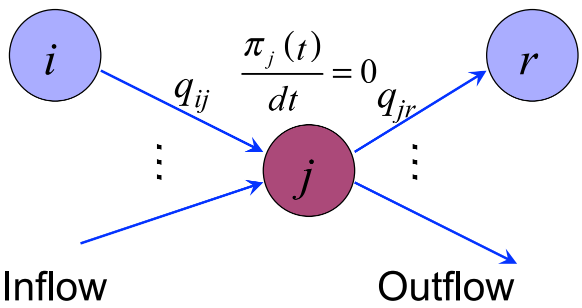
- Using the "probability fluid" view

outflow

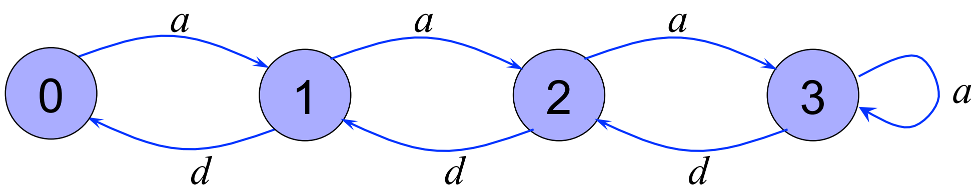
$$0 = q_{jj}\pi_j(t) + \sum_{i \neq j} q_{ij}\pi_i(t)$$

0 *Change*

inflow



Example



- For the previous example, with the above transition function, what are the steady state probabilities
- Solve

$$\boldsymbol{\pi}\mathbf{Q} = [\pi_0 \quad \pi_1 \quad \pi_2 \quad \pi_3] \begin{bmatrix} -\lambda & \lambda & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 \\ 0 & \mu & -(\lambda+\mu) & \lambda \\ 0 & 0 & \mu & -\mu \end{bmatrix} = \mathbf{0}$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$

Example

- The solution is obtained

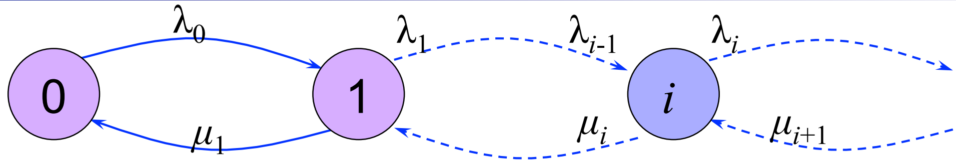
$$-\lambda\pi_0 + \mu\pi_1 = 0 \qquad \Rightarrow \pi_1 = \frac{\lambda}{\mu}\pi_0$$

$$\lambda\pi_0 - (\lambda+\mu)\pi_1 + \mu\pi_2 = 0 \qquad \Rightarrow \pi_2 = \left(\frac{\lambda}{\mu}\right)^2 \pi_0$$

$$\lambda\pi_1 - (\lambda+\mu)\pi_2 + \mu\pi_3 = 0 \qquad \Rightarrow \pi_3 = \left(\frac{\lambda}{\mu}\right)^3 \pi_0$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \Rightarrow \qquad \pi_0 = \frac{1}{1 + \left(\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3}$$

Birth-Death Chain



- Find the steady state probabilities
- Similarly to the previous example,

$$\mathbf{Q} = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & \cdots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- And we solve

$$\boldsymbol{\pi}\mathbf{Q} = \mathbf{0} \qquad \text{and} \qquad \sum_{i=0}^{\infty} \pi_i = 1$$

Example

- The solution is obtained

$$-\lambda_0 \pi_0 + \mu_1 \pi_1 = 0 \qquad \Rightarrow \pi_1 = \frac{\lambda_0}{\mu_1} \pi_0$$

$$\lambda_0 \pi_0 - (\lambda_1 + \mu_1)\pi_1 + \mu_2 \pi_2 = 0 \qquad \Rightarrow \pi_2 = \left(\frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} \right) \pi_0$$

- In general

$$\lambda_{j-1} \pi_{j-1} - (\lambda_j + \mu_j)\pi_j + \mu_{j+1} \pi_{j+1} = 0 \qquad \Rightarrow \pi_{j+1} = \left(\frac{\lambda_0 \ldots \lambda_j}{\mu_1 \ldots \mu_{j+1}} \right) \pi_0$$

- Making the sum equal to 1

$$\pi_0 \left(1 + \sum_{j=1}^{\infty} \left(\frac{\lambda_0 \ldots \lambda_{j-1}}{\mu_1 \ldots \mu_j} \right) \right) = 1$$

Solution exists if

$$S = 1 + \sum_{j=1}^{\infty} \left(\frac{\lambda_0 \ldots \lambda_{j-1}}{\mu_1 \ldots \mu_j} \right) < \infty$$

Uniformization of Markov Chains

- In general, discrete-time models are easier to work with, and computers (that are needed to solve such models) operate in discrete-time
- Thus, we need a way to turn continuous-time to discrete-time Markov Chains

- **Uniformization Procedure**
  - Recall that the total rate out of state i is $-q_{ii}=\Lambda(i)$. Pick a uniform rate γ such that $\gamma \geq \Lambda(i)$ for all states i.
  - The difference $\gamma - \Lambda(i)$ implies a "fictitious" event that returns the MC back to state i (self loop).

Uniformization of Markov Chains

- **Uniformization Procedure**
  - Let P^U_{ij} be the transition probability from state i to state j for the discrete-time uniformized Markov Chain, then

$$P^U_{ij} = \begin{cases} \dfrac{q_{ij}}{\gamma} & \text{if } i \neq j \\ \dfrac{\gamma - \sum_{j \neq i} q_{ij}}{\gamma} & \text{if } i = j \end{cases}$$