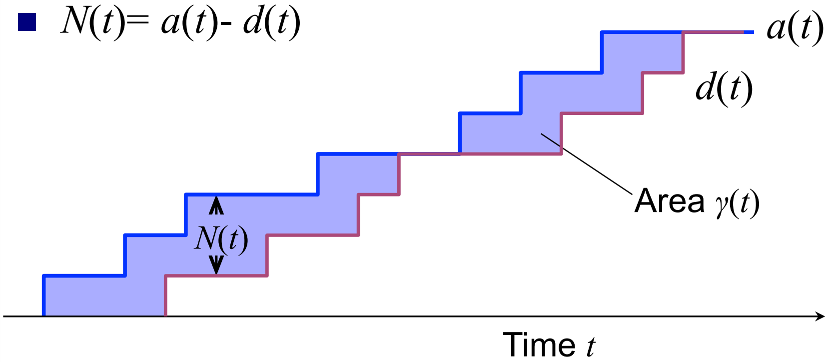
Queueing Theory I

Summary

- Little's Law
- Queueing System Notation
- Stationary Analysis of Elementary Queueing Systems
  - M/M/1
  - M/M/m
  - M/M/1/K
  - …

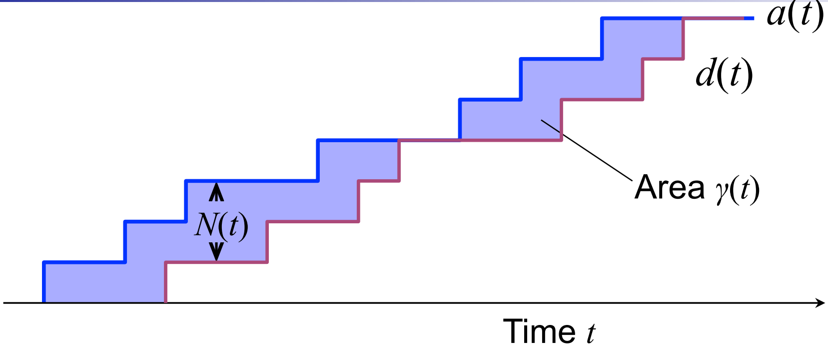
Little's Law

- $a(t)$: the process that counts the number of arrivals up to t.
- $d(t)$: the process that counts the number of departures up to t.
- $N(t)= a(t)- d(t)$



- Average arrival rate (up to t) $\lambda_t= a(t)/t$
- Average time each customer spends in the system $T_t= \gamma(t)/a(t)$
- Average number in the system $N_t= \gamma(t)/t$

Little's Law



$$N_t = \lambda_t T_t$$

- Taking the limit as t goes to infinity

$$E[N] = \lambda E[T]$$

Expected number of customers in the system

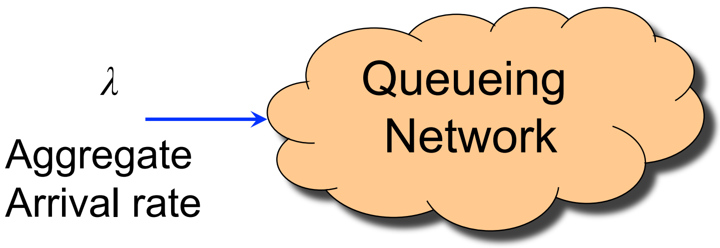
Expected time in the system

Arrival rate **IN** the system

Generality of Little's Law

$$E[N] = \lambda E[T]$$

- Little's Law is a pretty general result
- It does not depend on the arrival process distribution
- It does not depend on the service process distribution
- It does not depend on the number of servers and buffers in the system.



Specification of Queueing Systems

- Customer *arrival* and *service* stochastic models
- Structural Parameters
  - Number of servers
  - Storage capacity
- Operating policies
  - Customer class differentiation (are all customers treated the same or do some have priority over others?)
  - Scheduling/Queueing policies (which customer is served next)
  - Admission policies (which/when customers are admitted)

Queueing System Notation

A/B/m/K/N

Arrival Process
- **M**: Markovian
- **D**: Deterministic
- **Er**: Erlang
- **G**: General

Service Process
- **M**: Markovian
- **D**: Deterministic
- **Er**: Erlang
- **G**: General

Number of servers m=1,2,…

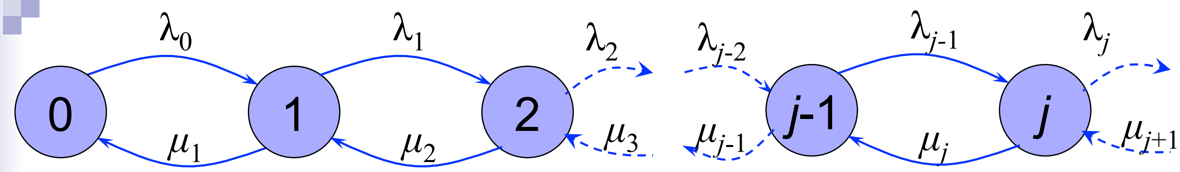
Storage Capacity K= 1,2,…
(if ∞ then it is omitted)

Number of customers N= 1,2,…
(for closed networks otherwise it is omitted)

Performance Measures of Interest

- We are interested in steady state behavior
  - Even though it is possible to pursue transient results, it is a significantly more difficult task.
- E[S] average system time (average time spent in the system)
- E[W] average waiting time (average time spent waiting in queue(s))
- E[X] average queue length
- E[U] average utilization (fraction of time that the resources are being used)
- E[R] average throughput (rate that customers leave the system)
- E[L] average customer loss (rate that customers are lost or probability that a customer is lost)

Recall the Birth-Death Chain Example



- At steady state, we obtain

$$-\lambda_0\pi_0 + \mu_1\pi_1 = 0 \qquad \Rightarrow \pi_1 = \frac{\lambda_0}{\mu_1}\pi_0$$

- In general

$$\lambda_{j-1}\pi_{j-1} - (\lambda_j + \mu_j)\pi_j + \mu_{j+1}\pi_{j+1} = 0 \qquad \Rightarrow \pi_{j+1} = \left(\frac{\lambda_0 \ldots \lambda_j}{\mu_1 \ldots \mu_{j+1}}\right)\pi_0$$

- Making the sum equal to 1

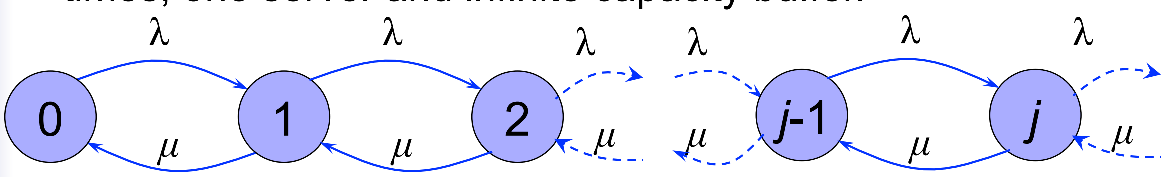
$$\pi_0\left(1 + \sum_{j=1}^{\infty}\left(\frac{\lambda_0 \ldots \lambda_{j-1}}{\mu_1 \ldots \mu_j}\right)\right) = 1$$

Solution exists if

$$S = 1 + \sum_{j=1}^{\infty}\left(\frac{\lambda_0 \ldots \lambda_{j-1}}{\mu_1 \ldots \mu_j}\right) < \infty$$

M/M/1 Example

- **Meaning**: Poisson Arrivals, exponentially distributed service times, one server and infinite capacity buffer.



- Using the birth-death result $\lambda_j=\lambda$ and $\mu_j=\mu$, we obtain

$$\pi_j = \left(\frac{\lambda}{\mu}\right)^j \pi_0, \quad j = 0,1,2,\ldots$$

- Therefore

$$\pi_0\left(1 + \sum_{j=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^j\right) = 1 \qquad \text{for } \lambda/\mu = \rho < 1$$

$$\pi_0 = 1 - \rho$$

$$\pi_j = (1-\rho)\rho^j, \quad j = 1,2,\ldots$$

M/M/1 Performance Metrics

- Server Utilization

$$E[U]=\sum_{j=1}^{\infty}\pi_j=1-\pi_0=1-(1-\rho)=\rho$$

- Throughput

$$E[R]=\mu\sum_{j=1}^{\infty}\pi_j=\mu(1-\pi_0)=\mu\rho=\lambda$$

- Expected Queue Length

$$E[X]=\sum_{j=0}^{\infty}j\pi_j=(1-\rho)\sum_{j=0}^{\infty}j\rho^j=\rho(1-\rho)\sum_{j=0}^{\infty}\frac{d\{\rho^j\}}{d\rho}=$$

$$=\rho(1-\rho)\frac{d}{d\rho}\left\{\sum_{j=0}^{\infty}\rho^j\right\}=\rho(1-\rho)\frac{d}{d\rho}\left\{\frac{1}{(1-\rho)}\right\}=\frac{\rho}{(1-\rho)}$$

M/M/1 Performance Metrics

- Average System Time

$$E[X]=\lambda E[S]\Rightarrow E[S]=\frac{1}{\lambda}E[X]$$

$$E[S]=\frac{1}{\lambda}\frac{\rho}{(1-\rho)}=\frac{1}{\mu(1-\rho)}$$

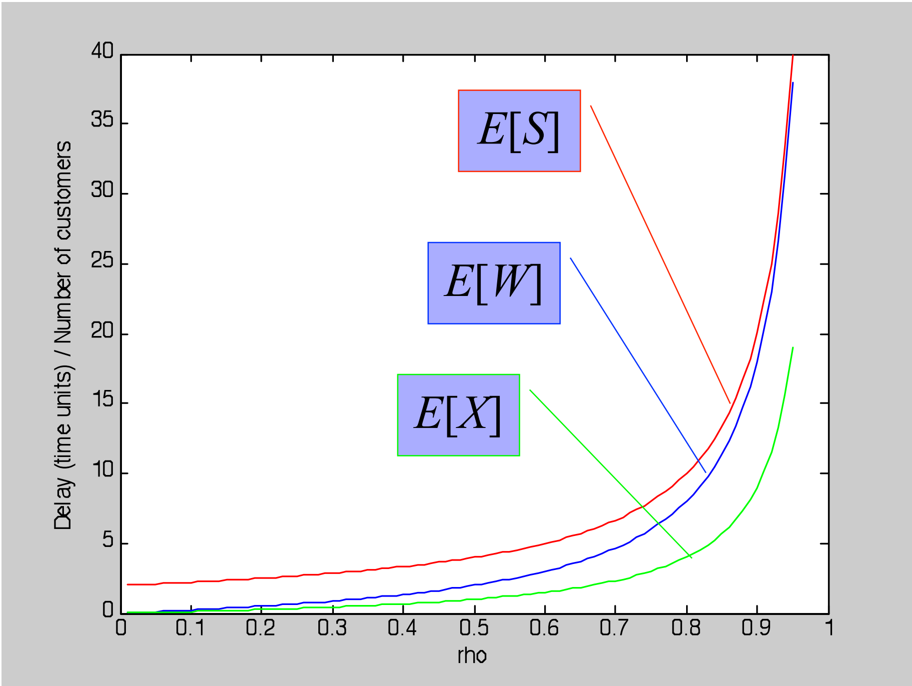
- Average waiting time in queue

$$E[S]=E[W]+E[Z]\Rightarrow E[W]=E[S]-E[Z]$$

$$E[W]=\frac{1}{\mu(1-\rho)}-\frac{1}{\mu}=\frac{\rho}{\mu(1-\rho)}$$

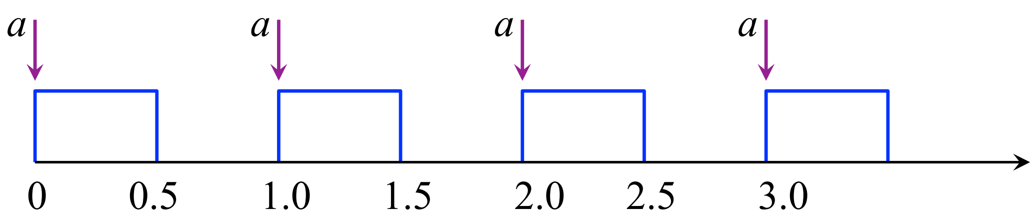
M/M/1 Performance Metrics Examples

- μ=0.5



PASTA Property

- PASTA: Poisson Arrivals See Time Averages
- Let $\pi_j(t)$= Pr{ System state $X(t)$= j }
- Let $a_j(t)$= Pr{ Arriving customer at t finds $X(t)$= j }
- In general $\pi_j(t) \neq a_j(t)$!
  - Suppose a D/D/1 system with interarrival times equal to 1 and service times equal to 0.5



- Thus $\pi_0(t)$= 0.5 and $\pi_1(t)$= 0.5 while $a_0(t)$= 1 and $a_1(t)$= 0!

Theorem

- For a queueing system, when the arrival process is Poisson and independent of the service process then, the probability that an arriving customer finds j customers in the system is equal to the probability that the system is at state j. In other words, $a_j(t) = \pi_j(t) \equiv \Pr\{X(t) = j\}, \quad j = 0, 1, \ldots$

- Proof:

Arrival occurs in interval $(t, \Delta t)$

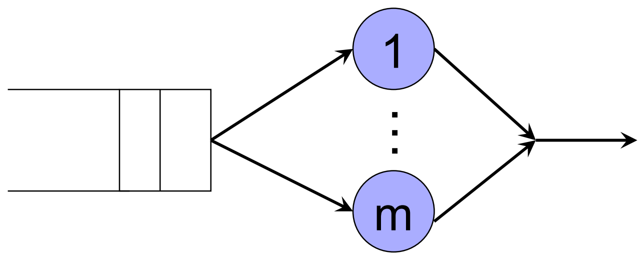
$$a_j(t) \equiv \lim_{\Delta t \to 0} \Pr\{X(t) = j \mid a(t, t+\Delta t)\}$$

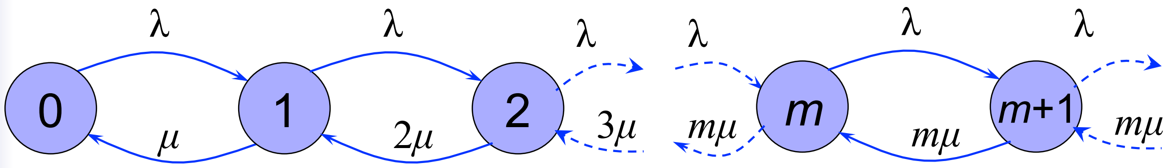
$$= \lim_{\Delta t \to 0} \frac{\Pr\{X(t) = j, a(t, t+\Delta t)\}}{\Pr\{a(t, t+\Delta t)\}}$$

$$= \lim_{\Delta t \to 0} \frac{\Pr\{X(t) = j\}\Pr\{a(t, t+\Delta t)\}}{\Pr\{a(t, t+\Delta t)\}} = \Pr\{X(t) = j\} = \pi_j(t)$$

M/M/m Queueing System

- **Meaning**: Poisson Arrivals, exponentially distributed service times, m identical servers and infinite capacity buffer.





$$\lambda_j = \lambda \quad \text{and} \quad \mu_j = \begin{cases} j\mu & \text{if } 0 \le j \le m \\ m\mu & \text{if } j \ge m \end{cases}$$

M/M/m Queueing System

- Using the general birth-death result

$$\pi_j = \frac{1}{j!}\left(\frac{\lambda}{\mu}\right)^j \pi_0, \quad \text{if } j < m \qquad \pi_j = \frac{m^m}{m!}\left(\frac{\lambda}{m\mu}\right)^j \pi_0, \quad \text{if } j \geq m$$

- Letting $\rho=\lambda/(m\mu)$ we get

$$\pi_j = \begin{cases} \dfrac{(m\rho)^j}{j!}\pi_0 & \text{if } j < m \\ \dfrac{m^m \rho^j}{m!}\pi_0 & \text{if } j \geq m \end{cases}$$

- To find π_0

$$\pi_0\left(1+\sum_{j=1}^{m-1}\frac{(m\rho)^j}{j!}+\sum_{j=m}^{\infty}\frac{m^m\rho^j}{m!}\right)=1 \Rightarrow \pi_0 = \left(1+\sum_{j=1}^{m-1}\frac{(m\rho)^j}{j!}+\frac{(m\rho)^m}{m!(1-\rho)}\right)^{-1}$$

M/M/m Performance Metrics

- Server Utilization

$$E[U]=\sum_{j=1}^{m-1} j\pi_j + m\Pr\{X\geq m\} = \pi_0\left(\sum_{j=1}^{m-1} j\frac{(m\rho)^j}{j!}+m\sum_{j=m}^{\infty}\frac{m^m\rho^j}{m!}\right)$$

$$=\pi_0\left((m\rho)+\sum_{j=2}^{m-1}\frac{(m\rho)^j}{(j-1)!}+m\frac{(m\rho)^m}{m!(1-\rho)}\right)$$

$$=\pi_0 m\rho\left(1+\sum_{j=2}^{m-1}\frac{(m\rho)^{j-1}}{(j-1)!}+\frac{(m\rho)^{m-1}}{(m-1)!}-\frac{(m\rho)^{m-1}}{(m-1)!}+\frac{m(m\rho)^{m-1}}{m!(1-\rho)}\right)$$

$$=\pi_0 m\rho\left(1+\sum_{j=1}^{m-1}\frac{(m\rho)^j}{j!}+\frac{(m\rho)^m}{m!(1-\rho)}\right)$$

$$=\pi_0 m\rho\frac{1}{\pi_0} = m\rho = \frac{\lambda}{\mu}$$

M/M/m Performance Metrics

- Throughput

$$E[R] = \mu \sum_{j=1}^{m-1} j\pi_j + m\mu \sum_{j=m}^{\infty} \pi_j = \lambda$$

- Expected Queue Length

$$E[X] = \sum_{j=0}^{\infty} j\pi_j = \pi_0 \left(\sum_{j=1}^{m-1} j \frac{(m\rho)^j}{j!} + \frac{m^m}{m!} \sum_{j=m}^{\infty} j\rho^j \right) = \ldots$$

$$E[X] = m\rho + \frac{(m\rho)^m}{m!} \frac{\rho}{(1-\rho)^2} \pi_0$$

- Using Little's Law

$$E[S] = \frac{1}{\lambda} E[X] = \frac{1}{\lambda} \left(m\rho + \frac{(m\rho)^m}{m!} \frac{\rho}{(1-\rho)^2} \pi_0 \right)$$

- Average Waiting time in queue

$$E[W] = E[S] - 1/\mu$$

M/M/m Performance Metrics

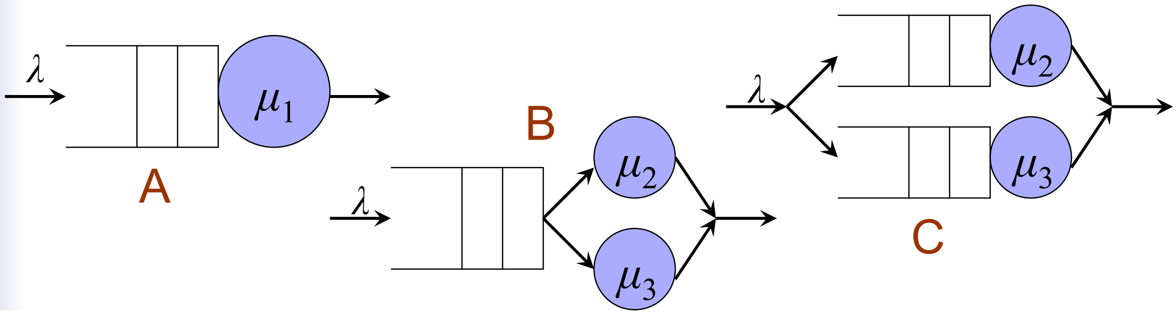
- Queueing Probability

$$P_Q = \Pr\{X \geq m\} = \sum_{j=m}^{\infty} \pi_j = \pi_0 \sum_{j=m}^{\infty} \frac{m^m \rho^j}{m!} = \frac{\pi_0 (m\rho)^m}{m!(1-\rho)}$$

Erlang C Formula

Example

- Suppose that packets arrive according to a Poisson process with rate $\lambda=1$. You are given the following two options,
  - Install a single transmitter with transmission capacity $\mu_1 = 1.5$
  - Install two identical transmitters with transmission capacity $\mu_2 = 0.75$ and $\mu_3 = 0.75$
  - Split the incoming traffic to two queues each with probability 0.5 and have $\mu_2 = 0.75$ and $\mu_3 = 0.75$ transmit from each queue.



Example

- Throughput
  - It is easy to see that all three systems have the same throughput $E[R_A] = E[R_B] = E[R_C] = \lambda$
- Server Utilization

$$E[U_A] = \frac{\lambda}{\mu_1} = \frac{1}{1.5} = \frac{2}{3}$$

$$E[U_B] = \frac{\lambda}{\mu_2} = \frac{1}{0.75} = \frac{4}{3}$$

Therefore, each server is 2/3 utilized

$$E[U_C] = \frac{0.5\lambda}{\mu_2} = \frac{1}{2\times 0.75} = \frac{2}{3}$$

  - Therefore, all transmitters are similarly loaded.

Example

- Probability of being idle

$$\pi_{0A} = 1 - \frac{\lambda}{\mu_1} = \frac{1}{3}$$

$$\pi_{0B} = \left(1 + \sum_{j=1}^{m-1} \frac{(m\rho)^j}{j!} + \frac{(m\rho)^m}{m!(1-\rho)}\right)^{-1} = \left(1 + \frac{4}{3} + \frac{\left(\frac{4}{3}\right)^2}{2\left(1-\frac{2}{3}\right)}\right)^{-1} = \frac{1}{5}$$

$$\pi_{0C} = 1 - \frac{\lambda}{2\mu_2} = \frac{1}{3}$$

For each transmitter

Example

- Queue length and delay

$$E[X_A] = \frac{\lambda}{\mu_1 - \lambda} = \frac{1}{1.5-1} = 2 \qquad E[S_A] = \frac{1}{\lambda} E[X_A] = 2$$

$$E[X_B] = m\rho + \frac{(m\rho)^m}{m!} \frac{\rho}{(1-\rho)^2} \pi_0 = \frac{12}{5} \qquad E[S_B] = \frac{1}{\lambda} E[X_B] = \frac{12}{5}$$

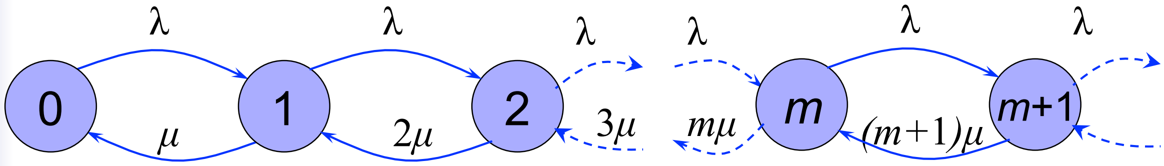
$$E[X_{1C}] = \frac{\lambda/2}{\mu_2 - \lambda/2} = \frac{0.5}{0.75-0.5} = 2$$

For each queue!

$$\Rightarrow E[X_C] = 2 \times E[X_{1C}] = 4 \qquad E[X_C] = \frac{1}{\lambda} E[X_C] = 4$$

M/M/∞ Queueing System

- Special case of the M/M/m system with m going to ∞



$$\lambda_j = \lambda \quad \text{and} \quad \mu_j = j\mu \quad \text{for all } j$$

- Let $\rho=\lambda/\mu$ then, the state probabilities are given by

$$\pi_j = \frac{\rho^j}{j!}\pi_0 \qquad \pi_0\left(1+\sum_{j=1}^{\infty}\frac{\rho^j}{j!}\right)=1 \Rightarrow \pi_0 = e^{-\rho} \qquad \Rightarrow \pi_j = \frac{\rho^j e^{-\rho}}{j!}$$

- System Utilization and Throughput

$$E[U]=1-\pi_0 = 1-e^{-\rho} \qquad E[R]=\lambda$$

M/M/∞ Performance Metrics

- Expected Number in the System

$$E[X]=\sum_{j=0}^{\infty} j\pi_j = \sum_{j=0}^{\infty} j\frac{\rho^j}{j!}e^{-\rho} = \rho e^{-\rho}\sum_{j=1}^{\infty}\frac{\rho^{j-1}}{(j-1)!} = \rho$$

Number of busy servers

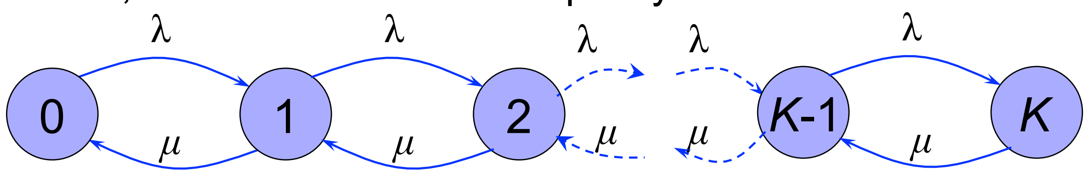
- Using Little's Law

$$E[S]=\frac{1}{\lambda}E[X]=\frac{1}{\lambda}\frac{\lambda}{\mu}=\frac{1}{\mu}$$

No queueing!

M/M/1/K – Finite Buffer Capacity

- **Meaning**: Poisson Arrivals, exponentially distributed service times, one server and finite capacity buffer K.



- Using the birth-death result $\lambda_j=\lambda$ and $\mu_j=\mu$, we obtain

$$\pi_j = \left(\frac{\lambda}{\mu}\right)^j \pi_0, \quad j = 0,1,2,\ldots K$$

- Therefore

$$\pi_0\left(1+\sum_{j=1}^{K}\left(\frac{\lambda}{\mu}\right)^j\right)=1$$

for $\lambda/\mu = \rho$

$$\pi_0 = \frac{1-\rho}{1-\rho^{K+1}}$$

$$\pi_j = \frac{(1-\rho)\rho^j}{1-\rho^{K+1}}, \quad j = 1,2,\ldots K$$

M/M/1/K Performance Metrics

- Server Utilization

$$E[U] = 1-\pi_0 = 1-\frac{(1-\rho)}{1-\rho^{K+1}} = \frac{\rho\left(1-\rho^K\right)}{1-\rho^{K+1}}$$

- Throughput

$$E[R] = \mu\left(1-\pi_0\right) = \lambda\frac{1-\rho^K}{1-\rho^{K+1}} < \lambda$$

- Blocking Probability

$$P_B = \pi_K = \frac{(1-\rho)\rho^K}{1-\rho^{K+1}}$$

Probability that an arriving customer finds the queue full (at state K)

M/M/1/K Performance Metrics

- Expected Queue Length

$$E[X]=\sum_{j=0}^{K} j\pi_j=\frac{(1-\rho)}{1-\rho^{K+1}}\sum_{j=0}^{K} j\rho^j=\frac{(1-\rho)\rho}{1-\rho^{K+1}}\sum_{j=0}^{K}\frac{d\{\rho^j\}}{d\rho}=$$

$$=\frac{(1-\rho)\rho}{1-\rho^{K+1}}\frac{d}{d\rho}\left\{\sum_{j=0}^{K}\rho^j\right\}=\frac{(1-\rho)\rho}{1-\rho^{K+1}}\frac{d}{d\rho}\left\{\frac{1-\rho^{K+1}}{(1-\rho)}\right\}=$$

$$=\frac{(1-\rho)\rho}{1-\rho^{K+1}}\left(\frac{(1-\rho^{K+1})-(1-\rho)(K+1)\rho^K}{(1-\rho)^2}\right)=$$

$$=\frac{\rho}{1-\rho^{K+1}}\left(\frac{1-\rho^K}{1-\rho}-K\rho^K\right)$$

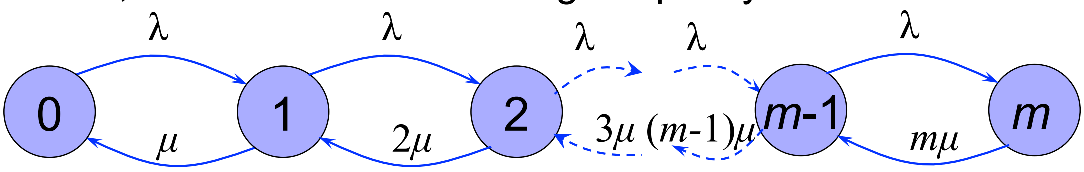
- System time

Net arrival rate (no losses)

$$E[X]=\lambda(1-\pi_K)E[S]$$

M/M/*m*/*m* – Queueing System

- **Meaning**: Poisson Arrivals, exponentially distributed service times, *m* servers and *no* storage capacity.



- Using the birth-death result $\lambda_j=\lambda$ and $\mu_j=\mu$, we obtain

$$\pi_j=\frac{1}{j!}\left(\frac{\lambda}{\mu}\right)^j\pi_0,\quad j=0,1,2,\ldots m$$

- Therefore

$$\pi_0\left(\sum_{j=0}^{m}\frac{1}{j!}\left(\frac{\lambda}{\mu}\right)^j\right)=1$$

for $\lambda/\mu=\rho$

$$\pi_0=\left(\sum_{j=0}^{m}\frac{\rho^j}{j!}\right)^{-1}$$

$$\pi_j=\frac{\rho^j}{j!}\pi_0,\quad j=1,2,\ldots m$$

M/M/*m*/*m* Performance Metrics

- Blocking Probability

$$P_B = \pi_m = \frac{\rho^m / m!}{\sum_{j=0}^{m} \frac{\rho^j}{j!}}$$

Erlang B Formula

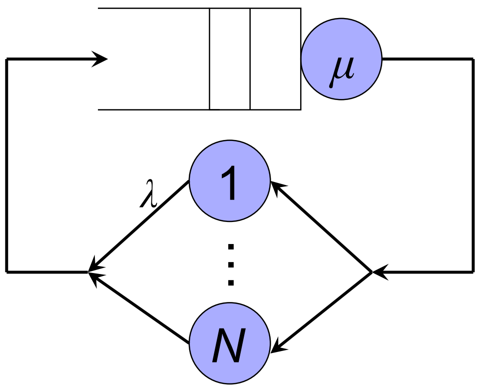
Probability that an arriving customer finds all servers busy (at state m)

- Throughput

$$E[R] = \lambda(1-\pi_m) = \lambda\left(1 - \frac{\rho^m / m!}{\sum_{j=0}^{m} \frac{\rho^j}{j!}}\right) < \lambda$$

M/M/1//N – Closed Queueing System

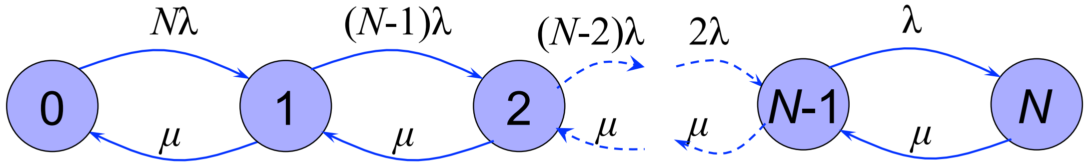
- **Meaning**: Poisson Arrivals, exponentially distributed service times, one server and the number of customers are fixed to *N*.



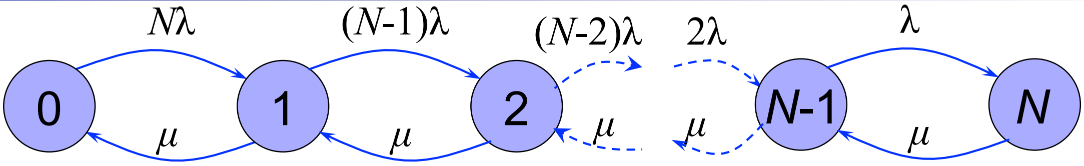
- Using the birth-death result, we obtain

$$\pi_j = \frac{N!}{(N-j)!}\rho^j \pi_0, \quad j = 1, 2, \ldots N$$

$$\pi_0 = \left[\sum_{j=0}^{N} \frac{N!}{(N-j)!}\rho^j\right]^{-1}$$



M/M/1//N – Closed Queueing System



- Response Time
  - Time from the moment the customer entered the queue until it received service.
- For the queue, using Little's law we get,

$$E[X]=\mu(1-\pi_0)E[S]$$

- In the "thinking" part,

$$E[N-X]=\mu(1-\pi_0)\frac{1}{\lambda}$$

- Therefore

$$E[S]=\frac{N-\mu(1-\pi_0)\frac{1}{\lambda}}{\mu(1-\pi_0)}=\frac{N}{\mu(1-\pi_0)}-\frac{1}{\lambda}$$