Digital Signal Processing

by

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February 2007 for University of Cyprus

Digital Signal Processing (DSP) Introduction

- Digital Signal Processing (DSP) is a branch of signal processing that emerged from the rapid development of VLSI technology that made feasible real-time digital computation.
- DSP involves time and amplitude quantization of signals and relies on the theory of discrete-time signals and systems.
- DSP emerged as a field in the 1960s.
- Early applications of off-line DSP include seismic data analysis, voice processing research.

Digital vs Analog Signal Processing

Advantages of digital over analog signal processing:

- flexibility via programmable DSP operations,
- storage of signals without loss of fidelity,
- off-line processing,
- lower sensitivity to hardware tolerances,
- rich media data processing capabilities,
- opportunities for encryption in communications,
- Multimode functionality and opportunities for software radio.

-Disadvantages :

- Large bandwidth and CPU demands

DSP Historical Perspective

- Nyquist Theorem 1920's.
- Statistical Time Series, PCM 1940's.
- Digital Filtering, FFT, Speech Analysis mid 1960s (MIT, Bell Labs, IBM).
- Adaptive Filters, Linear Prediction (Stanford, Bell Labs, Japan 1960s).
- Digital Spectral Estimation, Speech Coding (1970s).

DSP Historical Perspective (2)

- First Generation DSP Chips (Intel microcontroler, TI, AT&T, Motorola, Analog Devices (early 1980s)
- Low-cost DSPs (late 1980s)
- Vocoder Standards for civilian applications (late 1980s)
- Migration of DSP technologies in general purpose CPU/Controllers "native" DSP (1990s)
- High Complexity Rich Media Applications
- Low Power (Portable) Applications

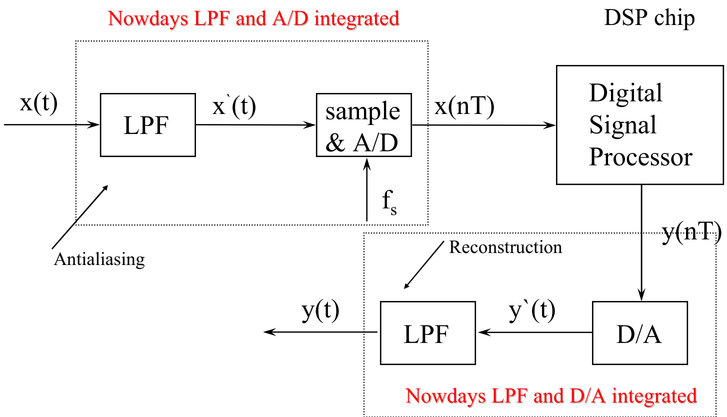
DSP Applications

- Military Applications (target tracking, radar, sonar, secure communications, sensors, imagery)
- Telecommunications (cellular, channel equalization, vocoders, software radioetc)
- PC and Multimedia Applications (audio/video on demand, streaming data applications, voice synthesis/recognition)
- Entertainment (digital audio/video compression, MPEG, CD, MD, DVD, MP3)
- Automotive (Active noise cancellation, hands-free communications, navigation-GPS, IVHS)
- Manufacturing, instrumentation, biomedical, oil exploration, robotics
- Remote sensing, security

Communications and DSP

- DTMF (use of the FFT and digital oscillators)
- Adaptive echo cancellation (Hands-free telephony, Speakerphones)
- Speech coding (speech coding in cellular phones)
- Modem (data/computer connectivity)
- Software radio (multi-mode/multi standard wireless communications)
- Channel estimation (equalization)
- Antenna beamforming (space division multiple access - SDMA)
- CDMA (modulating with random sequences)

Typical Digital Signal Processing System



Remarks: The diagram shows the sampling, processing, and reconstruction of an analog signal. There are applications where processing stops at the digital signal processor, e.g., speech recognition.

Symbols and Notation

$$x_a(t)|_{t=nT} = x(nT) = x(n) \quad ;\text{discrete}-\text{time input}$$

$$y(n) \quad ;\text{discrete}-\text{time output}$$

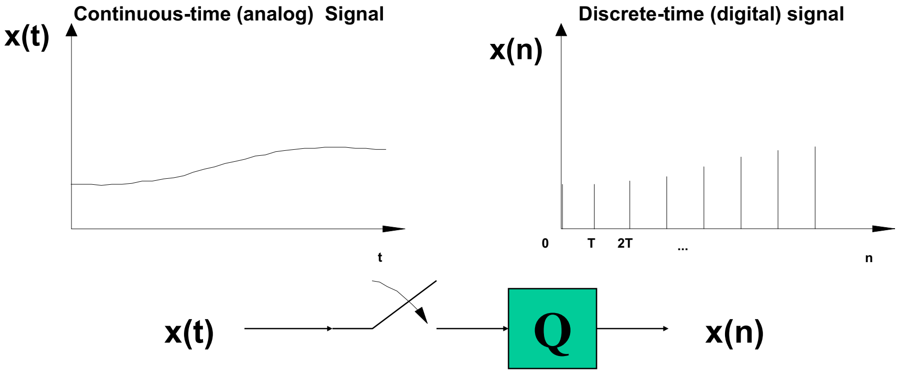
$$H(.) \quad ;\text{transfer and frequency response functions}$$

$$h(.) \quad ;\text{impulse response (system function)}$$

$$n \quad ;\text{discrete}-\text{time index}$$

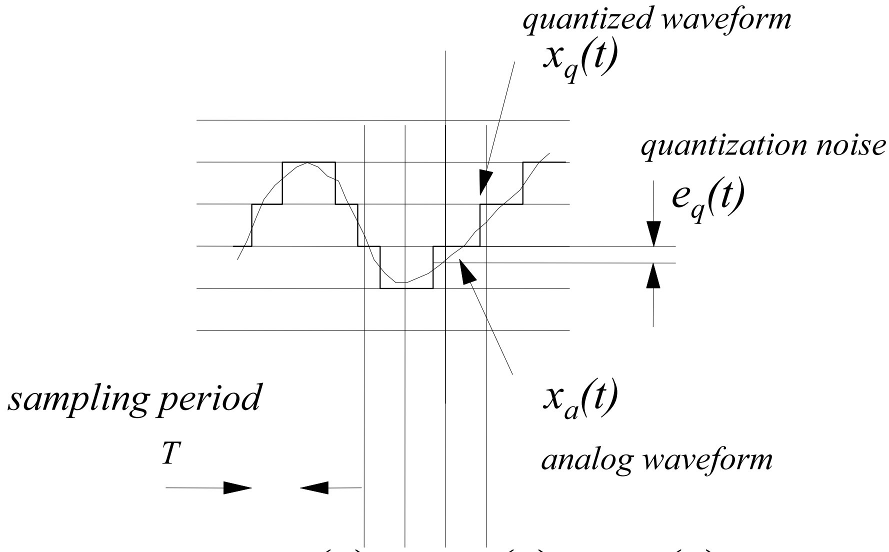
Remarks: In general and unless otherwise stated lower case symbols will be used for time-domain signals and upper case symbols will be used for transform domain signals. Bold face or underlined face symbols will be Be generally used for vectors or matrices.

Continuous vs Discrete-time



Remarks: A continuous-time signal is converted to discrete-time using sampling and quantization. As a result aliasing and quantization noise is introduced. This noise can be controlled by properly designing the quantizer and anti-aliasing filter.

Quantization Noise



$$x_q(t) = x_\alpha(t) + e_q(t)$$

Simplest Quantization Scheme - Uniform PCM

Performance in terms of Signal to Noise Ratio (SNR)

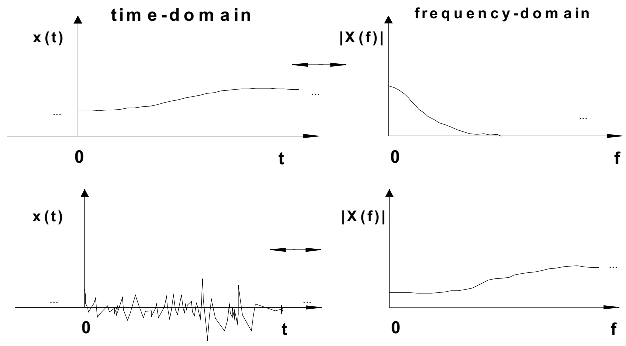
$$SNR_{PCM} = 6.02 R_b + K_1$$

where R_b is the number of bits and the value of K_1 depends on signal statistics. For telephone speech $K_1 = -10$

Oversampling Δ/Σ or Σ/Δ Conversion

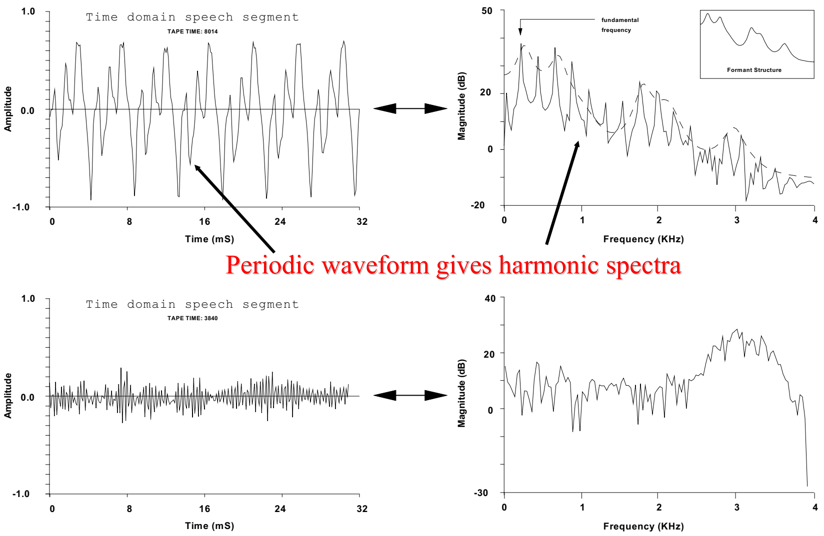
- Integrated oversampling and 1-bit quantization
- Very compact and inexpensive circuitry (some low power applications as well)
- Lowers analog circuit complexity with a modest increase in software (DSP MIPS) complexity
- Uses concepts from multirate signal processing and Delta Modulation
- Will be described in the context of multirate signal processing

Time vs Frequency Domain



Remarks: Slowly time-varying signals tend to have low-frequency content while signals with abrupt changes in their amplitudes have high frequency content. The frequency content of signals can be estimated using Fourier techniques.

Example: Time vs Frequency Domain Speech



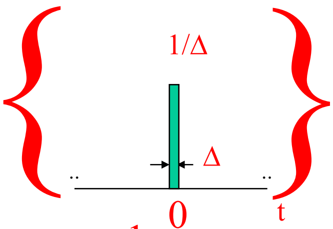
Review of Analog Signals and Systems

- FREQUENCY DOMAIN ANALYSIS
  - The Fourier series (measuring the spectrum of periodic signals)
  - The Fourier transform (measuring the spectrum of non-periodic signals and generally all signals)
- SAMPLING
  - The Sampling theorem (how we convert to digital signals without losing information)
- FILTERS
  - Continuous-time systems (analog filters)
  - Convolution (how filtering is done)

Some Important Signals

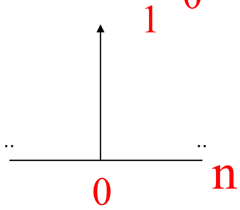
Continuous-time Impulse

$$\delta(t) = \lim_{\Delta \to 0}$$



Discrete-time Impulse

$$\delta(n) =$$

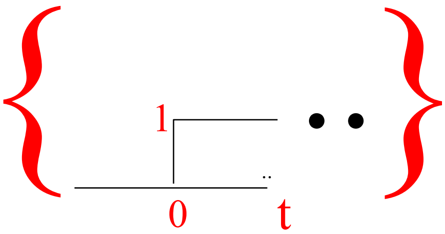


Think of signals as a sum of impulses.
Impulses help in analyzing signals and filters

Some Important Signals (2)

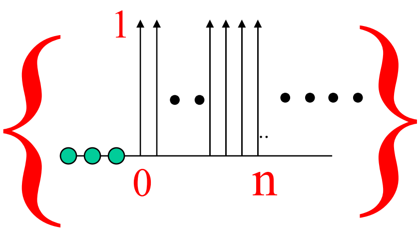
Continuous-time unit step

$$u(t) =$$



Discrete-time unit step

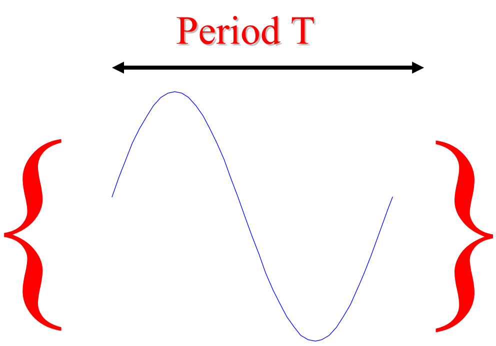
$$u(n) =$$



Some Important Signals (3)

The sinusoid

$$\sin(\omega t) = \sin(\frac{2\pi}{T}t) =$$



$$\omega = 2\pi f = \frac{2\pi}{T}$$

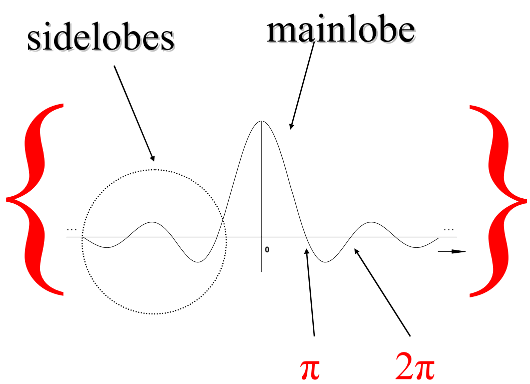
units: ω(rad/s) f(Hz) T(s)

Sinusoids are used in analyzing or synthesizing acoustic and other signals

Some Important Signals (4)

The sinc function

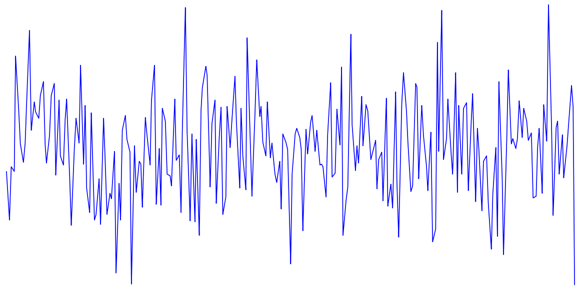
$$\mathrm{sinc}(t) = \frac{\sin(t)}{t} =$$



Sinc functions often appear in signal and filter analysis particularly when considering frequency domain behavior

Some Important Signals (5)

Random noise

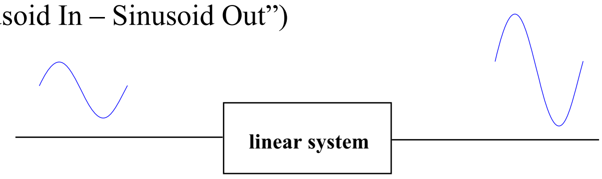


Encountered in communication systems and other application
Characterized by their mean and variance

Frequency-domain representations of signals

In order to observe and analyze the spectrum, the signal is usually represented in terms of other basic ("basis") signals. Basis signals or more precisely basis functions are typically chosen to be orthogonal. The most common orthogonal basis function used in signal analysis is the sinusoid. This is mainly because of:

- the physical properties of a sinusoid, i.e., as an acoustic tone
- the fact that sinusoids are "eigenfunctions" of linear systems ("Sinusoid In – Sinusoid Out")



Representing Periodic Signals with Sinusoids

Fourier series: Trigonometric form:

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_o t)$$

Fourier series: Complex (magnitude/phase) form:

Preferred in engineering -->> $$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_o t}$$

$\boldsymbol{X_k}$ are complex F.S. coefficients and provide spectral magnitude and phase info

and $$e^{jk\omega_o t} = \cos(k\omega_o t) + j\sin(k\omega_o t)$$

The Complex Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_o t}$$

Synthesis Expression

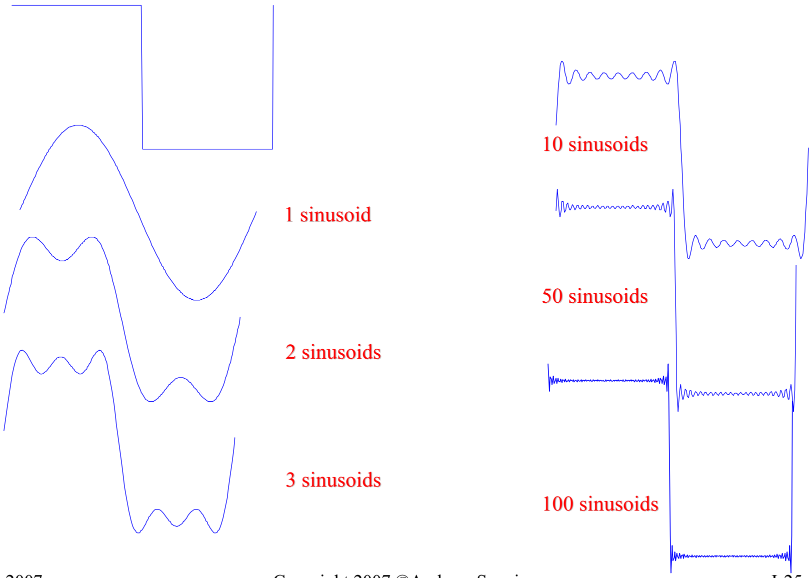
$$X_k = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_o t} dt$$

Analysis Expression
X_k are discrete F.S. spectral coefficients

where $\omega_o = 2\pi / T$

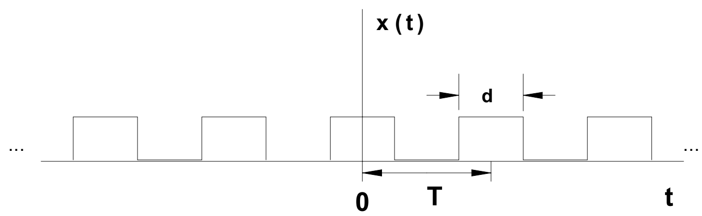
The magnitude of F.S. coefficients, $\boldsymbol{X_k}$, provides info on frequency content. Phase of $\boldsymbol{X_k}$ often provides info on events in signal (e.g., beginning of a period etc.)

Use Sinusoids to synthesize a periodic pulse using the Fourier series (only one period shown)



Fourier Series Analysis Example

Representing a Periodic Pulse Train as a Sum of Harmonic Sinusoids

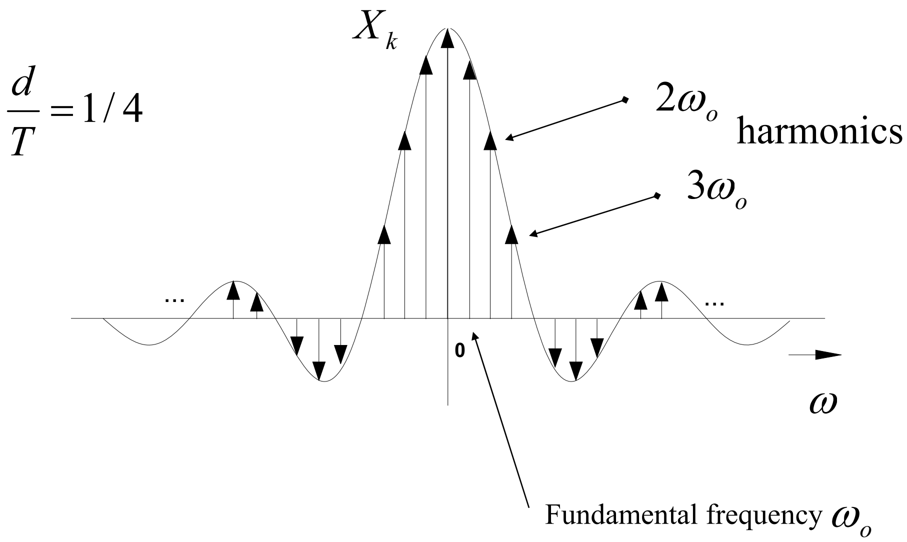


$$X_k = \frac{1}{T}\int_{-d/2}^{d/2} 1\, e^{-jk\omega_o t}\, dt = \frac{d}{T}\,\mathrm{sinc}\left(\frac{k\omega_o d}{2}\right)$$

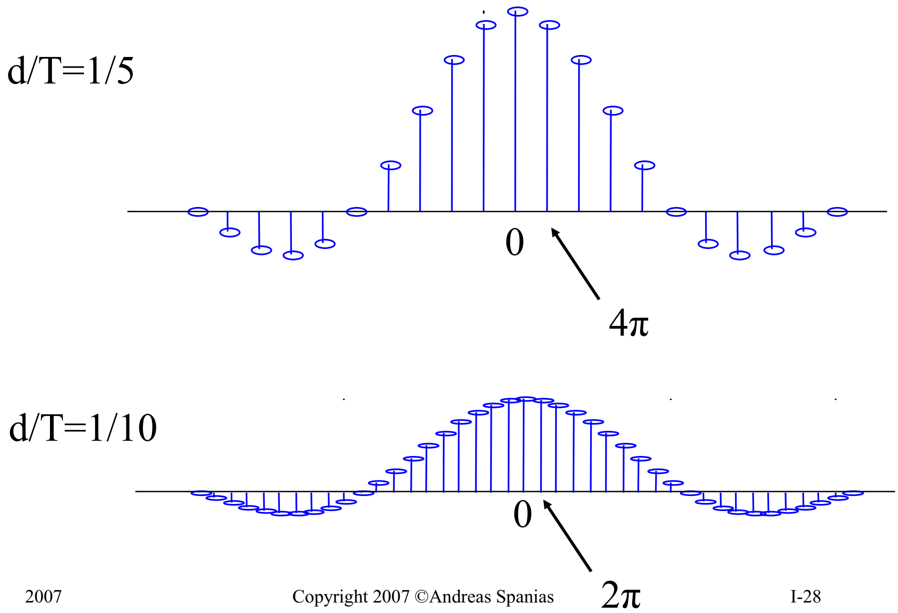
Remarks: A periodic pulse signal has a discrete F.S. spectrum described by samples that fall on a sinc (sinc(x)=sin(x)/x) function. As the period increases the F.S. components become more dense in frequency and weaker in amplitude. If T goes to infinity periodicity is lost and the F.S. vanishes.

Fourier Series Example (2)

Harmonic Spectrum



Fourier Series Example (3)



Remarks on the Fourier Series

- F.S. represents periodic signals with a sum of harmonic sinusoids
- the F.S.spectrum is discrete and F.S. components correspond to integer multiples of the fundamental frequency
- periodic signals have a discrete spectrum
- a uniformly sampled spectrum implies periodicity in the time domain
- A discrete-time F.S. is also available
- If $T \longrightarrow \infty$ the F.S. vanishes
- F.S. can be used for spectral analysis, filter design, and many other applications

Selected F.S. Properties

Linearity **(superposition holds – addition&scaling in time is addition&scaling in frequency)**

$$x(t) \overset{F.S.}{\longleftrightarrow} X_k \qquad y(t) \overset{F.S.}{\longleftrightarrow} Y_k$$

if $$z(t) = \alpha x(t) + \beta y(t)$$

then $$Z_k = \alpha X_k + \beta Y_k$$

Parseval's Theorem **(power is preserved from time to frequency)**

$$\sum_{k=-\infty}^{\infty} |X_k|^2 = \frac{1}{T} \int_{t_o}^{t_o+T} |x(t)|^2 \, dt$$

From the F.S. to the Continuous Fourier Transform

For non-periodic signals $T \to \infty \quad \& \quad \omega_o \to \Delta\omega$

$$x(t) = \lim_{\Delta\omega \to 0} \{ \frac{\Delta\omega}{2\pi} \sum_{k=-\infty}^{\infty} X_k \, e^{jk\Delta\omega t} \}$$

$$x(t) = \lim_{\Delta\omega \to 0} \{ \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} x(\tau) \, e^{jk\Delta\omega t \tau} \, d\tau \;\; e^{jk\Delta\omega t} \; \Delta\omega \}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} x(\tau) e^{j\omega\tau} d\tau] \;\; e^{j\omega t} \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega$$

Remarks: For non-periodic signals the F.S. vanishes. If the limit is taken then we can derive the continuous Fourier transform. The last equation is known as the inverse Fourier transform. Note that ω is now a continuous variable

The Continuous Fourier Transform (CFT) Equations

The Fourier transform — Analysis Expression

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

The inverse Fourier transform — Synthesis Expression

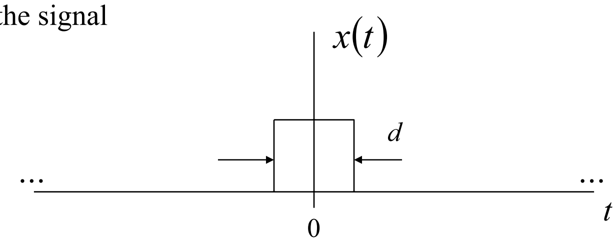
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

A Fourier transform pair is designated by: $x(t) \leftrightarrow X(\omega)$

Remarks: Both time and frequency are continuous variables. The CFT can handle non-periodic signals as long as they are integrable. Periodic signals can be handled using the impulse and CFT properties.

Fourier transform of a time-limited pulse

Given the signal

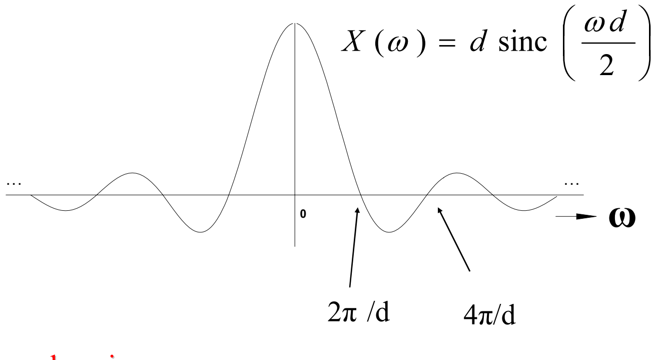


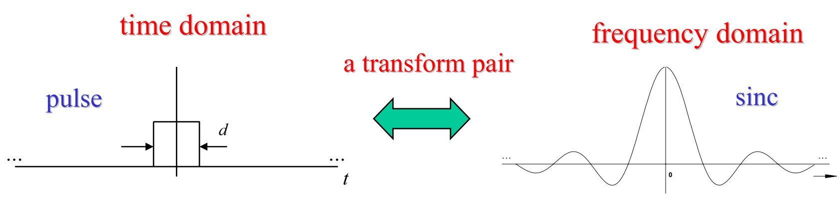
$$X(\omega) = \int_{-d/2}^{d/2} e^{-j\omega t}\, dt = d \,\mathrm{sinc}\left(\frac{\omega d}{2}\right)$$

Remarks: Note that a time-limited signal has a non-bandlimited CFT spectrum. The sinc function has zero crossings at integer multiples of $2\pi/d$. As the pulse width increases the sinc function "shrinks". In the limit, if T goes to infinity (i.e., pulse becomes D.C. signal) the sinc function collapses to a unit impulse.

Fourier transform of a time-limited pulse(Cont.)

$$X(\omega) = d \,\mathrm{sinc}\left(\frac{\omega d}{2}\right)$$



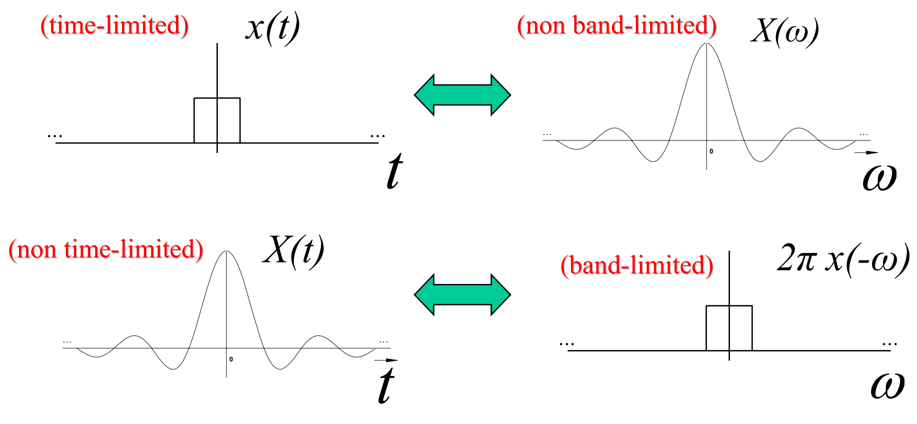


Remarks on the CFT

- Proper computation of the CFT spectrum requires that the signal is known everywhere in time
- Spectra of truncated signals suffer from spectral leakage and loss of resolution
- A time-limited signal has a non-band-limited CFT spectrum
- A band-limited signal can not be time-limited
- The forward and inverse CFT formulas are symmetric and therefore we observe dualities in CFT transform pairs and CFT properties
- Numerical computation of the CFT is done using the fast Fourier Transform (FFT)

Symmetry of the Fourier transform

$$\text{if} \quad x(t) \leftrightarrow X(\omega) \quad \text{then} \quad X(t) \leftrightarrow 2\pi x(-\omega)$$

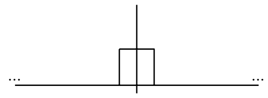


Selected F.T properties - Linearity

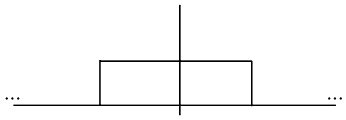
if $x(t) \leftrightarrow X(\omega)$ & $y(t) \leftrightarrow Y(\omega)$

then $\alpha x(t) + \beta y(t) \leftrightarrow \alpha X(\omega) + \beta Y(\omega)$

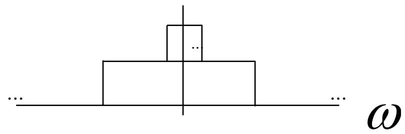
Example: $x(t) \Leftrightarrow$



$y(t) \Leftrightarrow$



$x(t) + y(t) \Leftrightarrow$



Selected F.T properties - Scaling

stretching a signal in time implies compressing it in frequency

compressing a signal in time implies stretching it in frequency

$$x(at) \leftrightarrow \frac{1}{|a|} X(\frac{\omega}{a})$$

Note that time expansion implies frequency compression

Selected F.T Properties - Time Shift

if $x(t) \leftrightarrow X(\omega)$

then $x(t-t_0) \leftrightarrow e^{-j\omega t_0} X(\omega)$

-linear phase factor
-has unit magnitude
- phase is linear across frequency

<u>Remarks</u>: A time shift introduces linear phase in the frequency domain

Selected F.T Properties – Frequency Shift

MODULATION - VERY IMPORTANT IN WIRELESS COMMUNICATIONS

If $x(t) \leftrightarrow X(\omega)$

then $e^{j\omega_0 t} x(t) \leftrightarrow X(\omega-\omega_0)$

multiplication by sinusoid translates the signal in frequency

The Time-Domain Convolution (Filtering) Property

$$x(t) \leftrightarrow X(\omega)$$

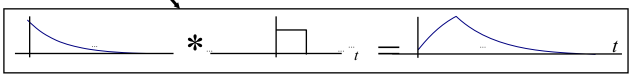
$$h(t) \leftrightarrow H(\omega)$$

$$h(t) * x(t) \leftrightarrow H(\omega)X(\omega)$$

$$h(t) * x(t) = \int h(\tau)x(t-\tau)d\tau$$

Muliplication in frequency
is essentially a filtering operation

convolution in time is multiplication in frequency



DEMO

Example: Convolution of an exponential with a pulse

Frequency Convolution

$$x(t) \leftrightarrow X(\omega)$$

$$w(t) \leftrightarrow W(\omega)$$

then

$$h(t)w(t) \leftrightarrow \frac{1}{2\pi} W(\omega) * X(\omega)$$

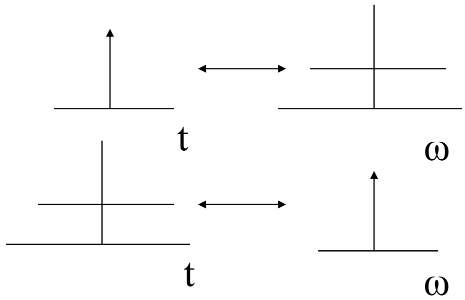
If w(t) is time limited
then this operation truncates

Convolution tends to have
a spreading effect

Important Fourier Transform Pairs

$$\delta(t) \leftrightarrow 1$$

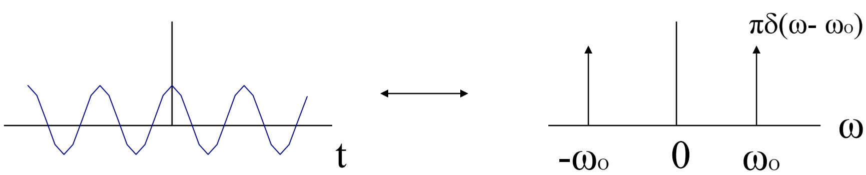
$$1 \leftrightarrow 2\pi\delta(\omega)$$



$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$$

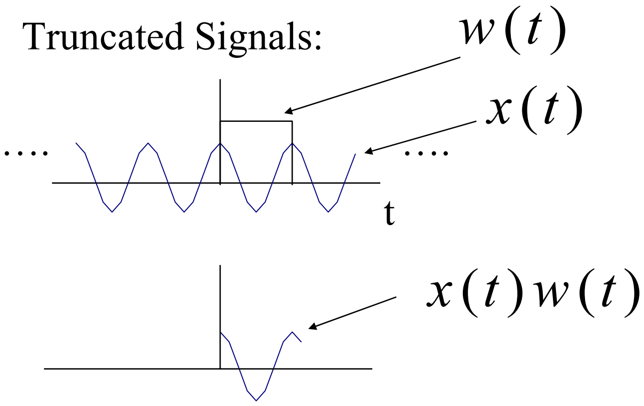
Important Fourier Transform Pairs (2)

$$\cos(\omega_0 t) \leftrightarrow \pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$$



Frequency Convolution and Windowing
Truncation

Truncated Signals:

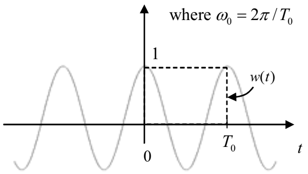


Remarks: Truncating an infinite-length signal is equivalent to multiplying it with a finite-length window. Multiplying a sinusoid with a rectangular pulse results in a finite-length sinusoid. All real-life signals are finite length.

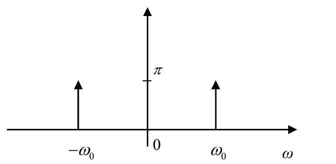
Truncating a Cosine

$s(t)=\cos(\omega_0 t)$ where $\omega_0 = 2\pi / T_0$

$S(\omega)=\pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))$



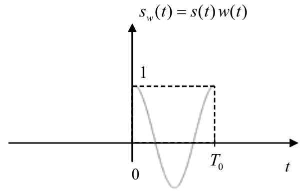
CFT ⟷



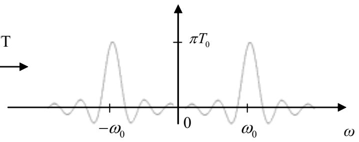
$$w(t)=\begin{cases}1; & 0 \le t \le T_0\\ 0; & \text{otherwise}\end{cases} \quad \overset{\text{CFT}}{\longleftrightarrow} \quad W(\omega)=T_0 e^{-j\omega T_0/2}\,\text{sinc}\left(\frac{\omega T_0}{2}\right)$$

$s_w(t)=s(t)\,w(t)$

$S_w(\omega)=\pi T_0 e^{-j\omega T_0/2}\left[\text{sinc}\left((\omega-\omega_0)T_0/2\right)+\text{sinc}\left((\omega+\omega_0)T_0/2\right)\right]$

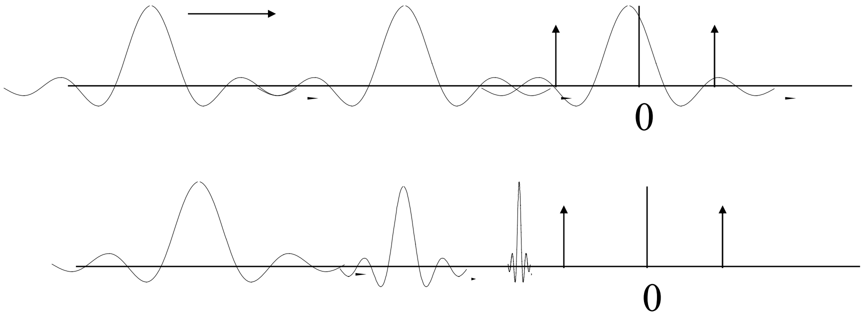


CFT ⟷



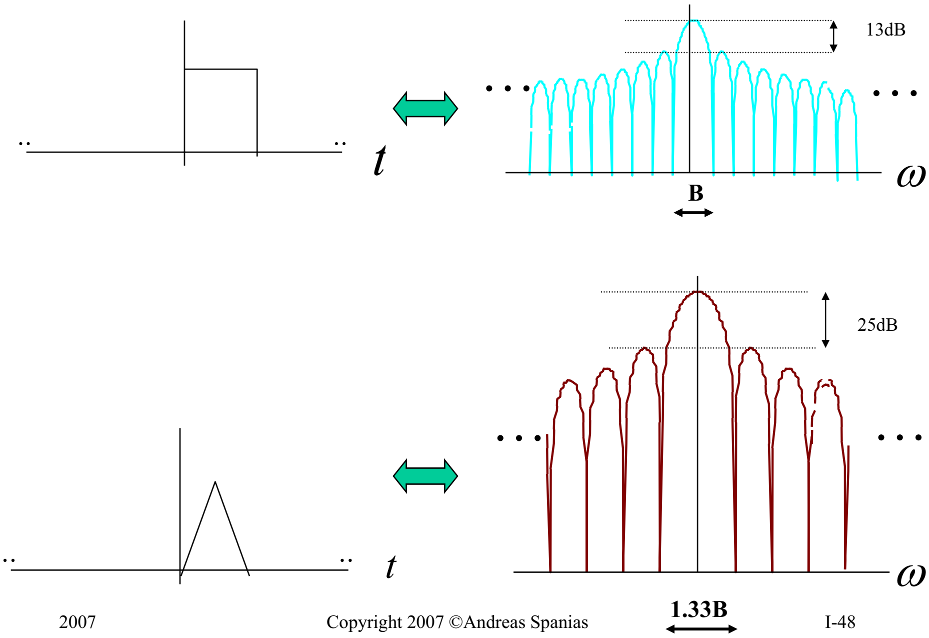
Truncating in time implies convolution in frequency

Truncation with a short rectangular window implies convolution with a "wideband" sinc

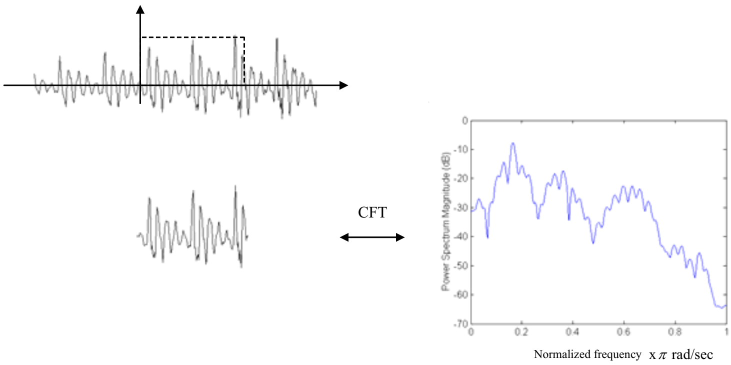


Remark: Truncation with a wider rectangular window implies convolution with a "narrowband" *sinc*. If the length of the rectangular window becomes arbitrarily long the sinc collapses to an impulse. Clearly, longer windows in time imply improved spectral resolution and less spectral leakage.

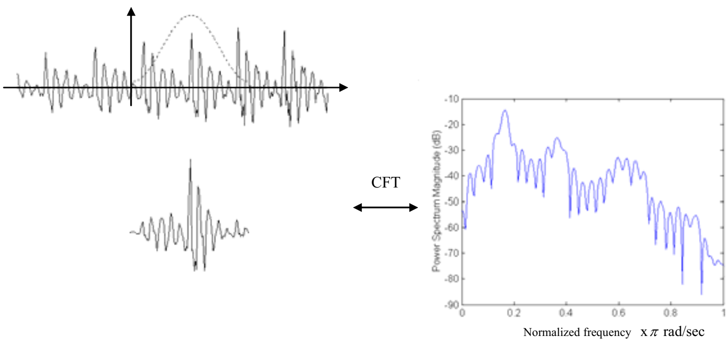
Truncating Signals with Tapered Windows



Truncating Speech



Truncating Speech (tapered window)



Truncating Signals with Tapered Windows (2)

•Truncation of a signal is inevitable in real life spectral estimation. Strong sidelobes contribute to spectral leakage and spectral smearing. The width of the mailobe affects the resolution of spectral estimates.

•In choosing a window one is confronted with the tradeoff of the mainlobe width and sidelobe level. Tapered windows have suppressed sidelobes relative to a rectangular window. However, the mainlobe width of a tapered window is wider than that of a rectangular window

•Truncation is also involved in designing FIR digital filters from Fourier series components. The sidelobe level affects the rejection and ripple characteristics while the mainobe width affects the transition characteristics.

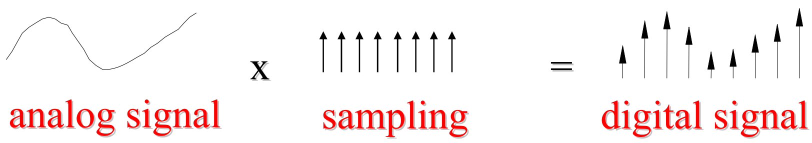
The Sampling Process

A bandlimited signal that has no spectral components at or above B can be uniquely represented by its sampled values spaced at uniform intervals that are not more than π/B seconds apart.

$$T \leq \frac{\pi}{B}$$

or a signal that is bandlimited to B must be sampled at a rate of ω_s where

$$\omega_s \geq 2B \quad or \quad f_s \geq \frac{B}{\pi}$$



Example: Audio - Bandwidth

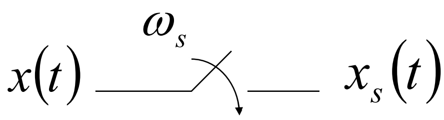
200- 3200 Hz	Basic Telephone Speech Intelligible Preserves Speaker Identity
50- 7000 Hz	Wideband Speech AM-grade audio
50- 15000 Hz	Near High Fidelity FM-grade Audio
20- 20000 Hz	High-Fidelity CD/DAT Quality Voice

Example: Sampling of Audio Signals

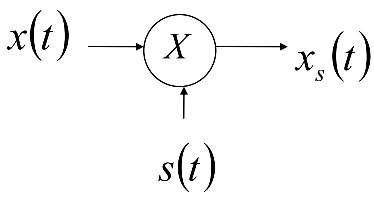
Format	Bandwidth	Sampling frequency
Telephony	3.2 kHz	8 kHz
Wideband audio	7 kHz	16 kHz
High-fidelity, CD	20 kHz	44.1 kHz
Digital audio tape (DAT)	20 kHz	48 kHz
Super audio CD (SACD)	100 kHz	2.8224 MHz
DVD audio (DVD-A)	96 kHz	192 kHz

The Math Representation of Sampling

Engineering representation



Mathematical representation

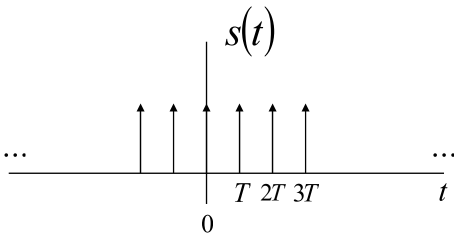


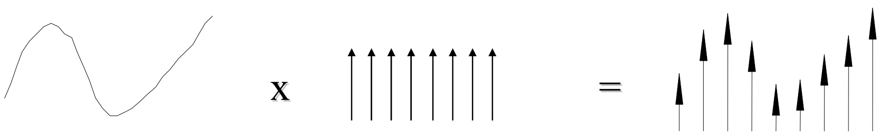
Remark: Multiplication with the ideal switching function results in a periodic spectrum where the signal spectrum is repeated at integer multiples of the sampling frequency.

Sampling by multiplying with impulses

The switching or sampling function

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



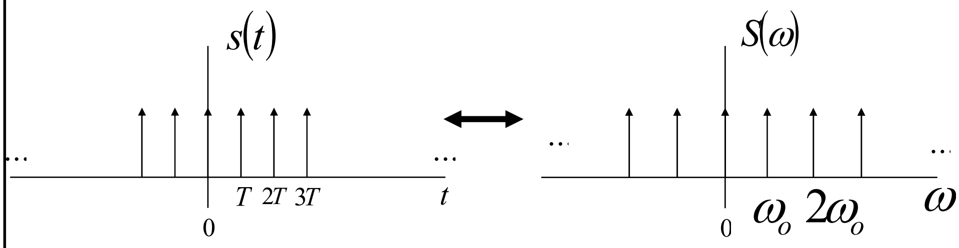


analog signal sampling digital signal

The Sampling Signal in the Frequency Domain

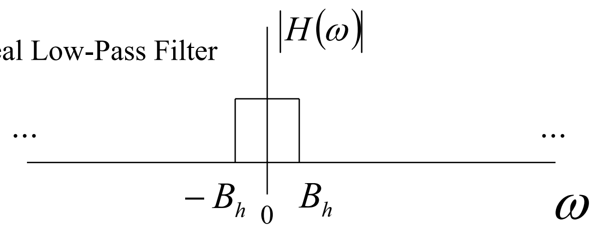
It can be easily shown that

$$\sum_{k=-\infty}^{\infty} \delta(t - nT) \leftrightarrow \omega_0 \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

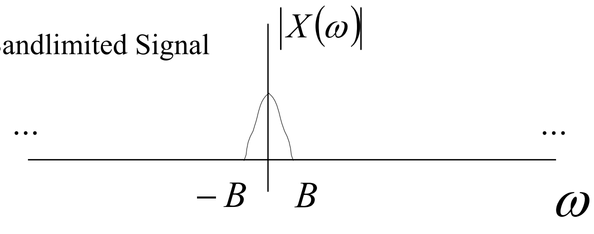


Ideal Low-Pass filters and Bandlimited Signals

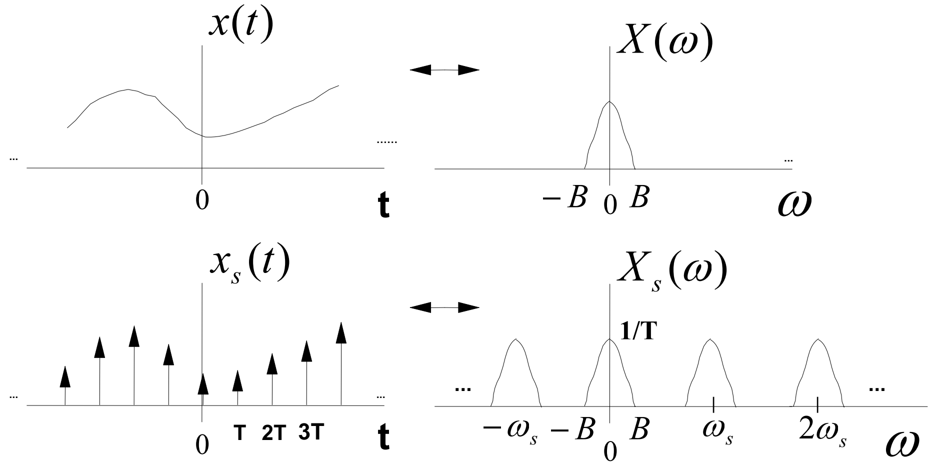
Ideal Low-Pass Filter



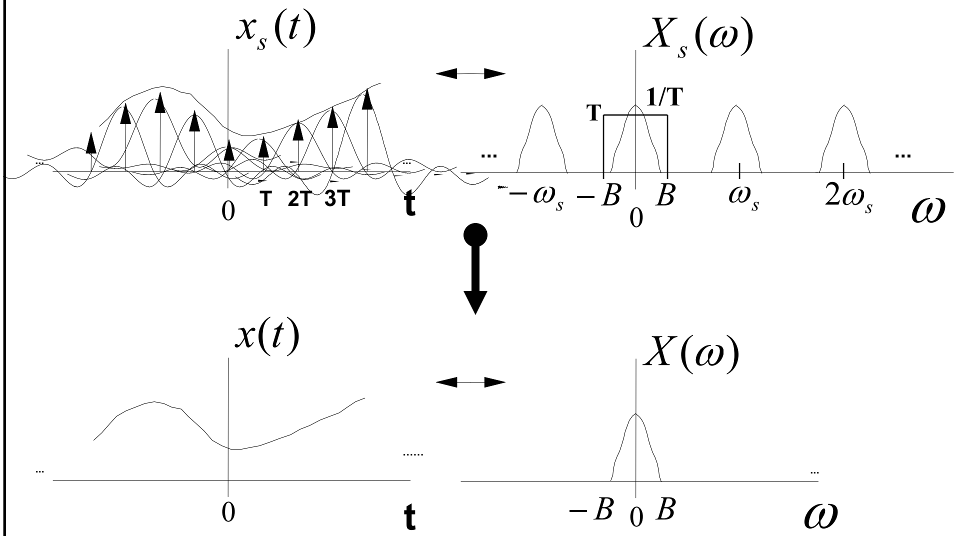
Ideally Bandlimited Signal



Sampling and Periodic Spectra



Signal Reconstruction using an Ideal Filter



Derivation of the Sampling Theorem

$$x_s(t) = x(t)s(t) \quad \text{where} \quad s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

and

$$X_s(\omega) = \frac{1}{2\pi} X(\omega) * S(\omega)$$

$$X_s(\omega) = \frac{1}{T} X(\omega) * \left\{ \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_o) \right\}$$

$$\Rightarrow \quad X_s(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_o)$$

Signal Reconstruction Analytically for ω_s=2B

$$h(t) * x_s(t) \leftrightarrow H(\omega) X_s(\omega)$$

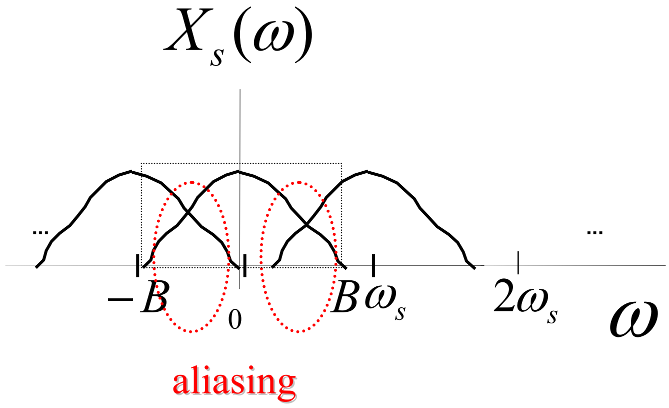
$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{j\omega t} d\omega = \text{sinc}(Bt)$$

$$x(t) = \text{sinc}(Bt) * \left\{ \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \right\}$$

$$\Rightarrow x(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\text{sinc}(B(t - nT))$$

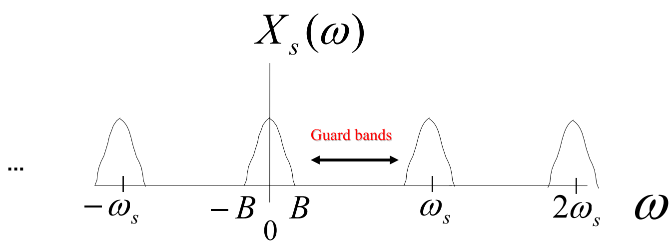
Remark: Note that the reconstruction filter *interpolates* between the samples with sinc functions - hence the name interpolation filter.

ALIASING (UNDERSAMPLING) $\omega_s<2B$



the signal can not be recovered perfectly even with an ideal filter
only a distorted version of the signal can be recovered

Oversampling $\omega_s>>2B$



Oversampling relaxes the requirements on antialiasing filters

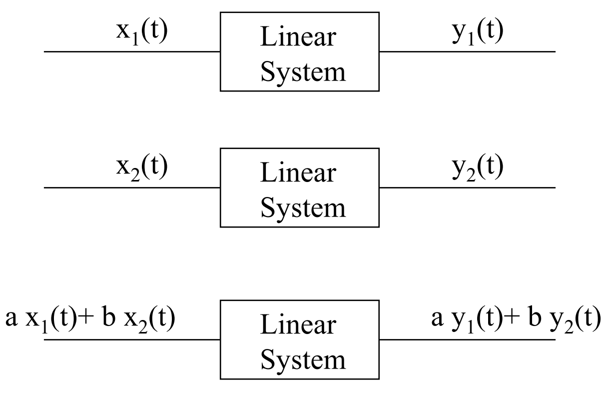
It is used in Σ/Δ (Δ/ Σ) A-to-D converters

Non-ideal Considerations of Sampling

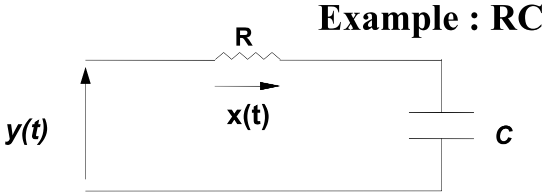
- There are no ideal impulses in practice - instead finite amplitude and finite duration periodic pulses are used (not a big problem)
- All real-life signals are not band-limited
- There are no ideal LPF
- Aliasing is always present and can be viewed as noise in the signal
- Typically we use antialiasing filters that limit aliasing noise several 10s of dBs under the useful signal energy
- Practical Rule: Quantization noise reduces generally 6 dB per added bit of resolution, e.g., at 16 bits we have approximately 96 dB SNR. Therefore we could built an antialiasing filter that keeps aliasing noise under the quantization noise by at least 6 dB.
- Bandpass signals can be sampled more efficiently

Continuous Linear Systems

In a linear system if we superimpose two distinct input signals $x_1(t)$ and $x_2(t)$ we get an output that consists of the superposition of the responses to each individual input, i.e.,



Example : RC Circuit



Assuming zero initial conditions

$$y_1(t) = R\ x_1(t) + \frac{1}{c}\int_0^1 x_1(\tau)\, d\tau$$

$$y_2(t) = R\ x_2(t) + \frac{1}{c}\int_0^1 x_2(\tau)\, d\tau$$

Note that: $\Rightarrow y(t) = \alpha y_1(t) + \beta y_2(t)$

Remarks on Continuous Linear Systems

•Analysis of continuous linear systems (CLS) relies on the theory of linear differential equations (LDE).

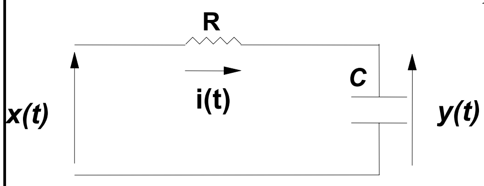
•The transient and steady-state responses of a CLS are obtained from the homogeneous and particular solutions respectively of the LDE.

•The output of a system can be obtained from the convolution integral of its impulse response convolved with the input

•The frequency response of a linear system is defined as the steady-state response to a spectrum of sinusoids.

•The steady-state output of a linear system due to a sinusoidal input is a sinusoid of the same frequency but phase shifted and amplitude scaled.

Example



Differential equation:

$$\frac{dy(t)}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} x(t)$$

Transfer function:

$$H(s) = \frac{1}{1+sRC}$$

Frequency response function:

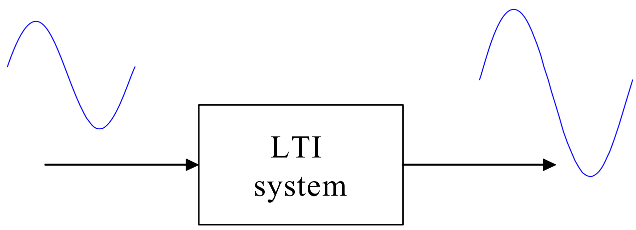
$$H(\omega) = \frac{1}{1+j\omega RC}$$

if $x(t) = \sin(\omega t)$ ← sinusoid in

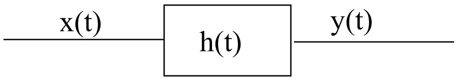
then

$$y^{ss}(t) = |H(\omega)| \sin(\omega t + \angle H(\omega))$$ ← sinusoid out

Sin In / Sin Out



Continuous Linear Systems (Cont.)



The output is obtained by convolving the input x(t) and the impulse response h(t) of the system, that is:

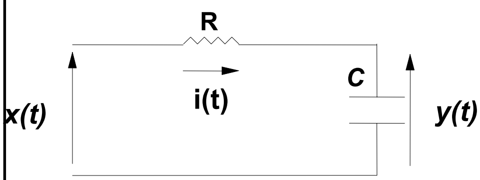
$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = h(t) * x(t)$$

For a causal system and causal input: <u>DEMO</u>

$$y(t) = \int_{0}^{t} h(\tau)x(t-\tau)d\tau$$

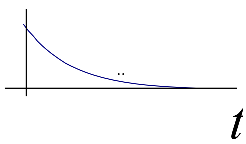
Example - Impulse Response

Consider the circuit below with R=1M, C=1x10^{-6}



$$\frac{dh(t)}{dt} + \frac{1}{RC}h(t) = \frac{1}{RC}\delta(t)$$

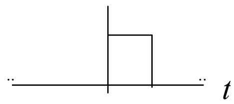
The solution: $h(t) = \frac{1}{RC}e^{-\frac{t}{RC}} = e^{-t}$ for $t \geq 0$



Example - Convolve and obtain an output

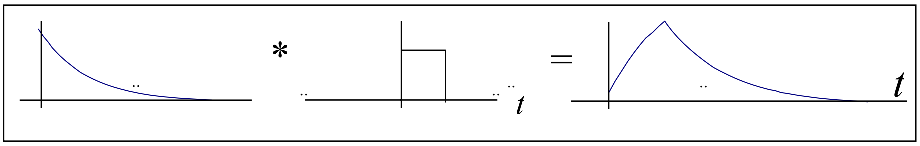
Consider the RC with impulse response $h(t) = e^{-t} u(t)$

and the input $x(t) = u(t) - u(t-1)$



$$y(t) = \int_0^t e^{-\tau} d\tau = 1 - e^{-t} \quad \text{for} \quad 0 < t < 1$$

$$y(t) = \int_{t-1}^t e^{-\tau} d\tau = -e^{-t} + e^{-(t-1)} \quad \text{for} \quad t > 1$$



Convolution of Pulses