The Z-Transform in DSP
Lecture 10-12

Andreas Spanias
spanias@asu.edu

Poles and Zeros of H(z)

In general the transfer function is rational; it has a numerator and a denominator polynomial.

The roots of the numerator and denominator polynomials are called the zeros and the poles respectively.

Pole-zero decompositions of *H(z)* are quite useful and provide intuition in signal analysis and filter design.

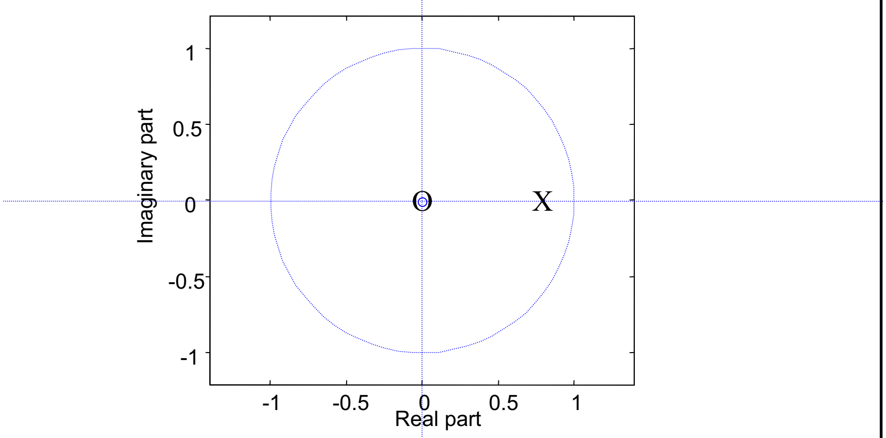
$$H(z) = G\frac{(z-\zeta_1)(z-\zeta_2)...(z-\zeta_L)}{(z-p_1)(z-p_2)...(z-p_M)} = G\frac{\prod_{i=1}^{L}(z-\zeta_i)}{\prod_{i=1}^{M}(z-p_i)}$$

where G is a gain factor

Example: Poles and Zeros of H(z)

$$H(z) = \frac{0.2z}{z-0.8}$$

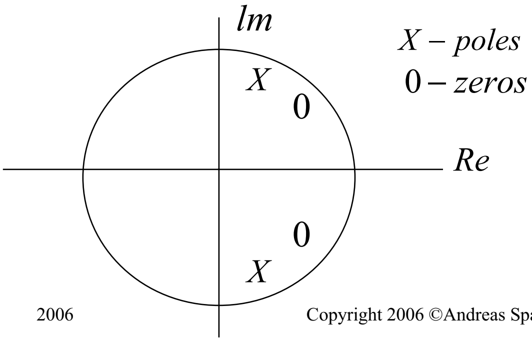
- the zero (O) of this transfer function is at z=0
- the pole (X) of this transfer function is at z=0.8



Example: Poles-Zeros of a Second Order System

$$H(z) = \frac{1 - 1.3435\, z^{-1} + 0.9025\, z^{-2}}{1 - 0.45\, z^{-1} + 0.55\, z^{-2}}$$

$$H(z) = \frac{(z-.95e^{j45^o})(z-.95e^{-j45^o})}{(z-.7416e^{j72.34^o})(z-.7416e^{-j72.34^o})}$$



Note that the filter coefficients are real valued and therefore poles and zeros occur in complex conjugate pairs.

Poles and Zeros and Stability

The location of the poles determines the stability of the filter. In fact, for stability all the poles of a causal filter must be inside the unit circle, that is

$$|p_i| < 1 \qquad \textit{for all} \quad i = 1, 2, \ldots, M$$

IIR filters may be all-pole or pole-zero and stability is always a concern. FIR or all-zero* filters are always stable.

(* depending on the way the transfer function is normalized the term all-zero for FIR systems may be imprecise because there may be poles located at zero in an FIR system - similarly an "all-pole" IIR may have zeros located at zero)

The Frequency Response Function

The transfer function is

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_L z^{-L}}{1 + a_1 z^{-1} + \ldots + a_M z^{-M}}$$

by evaluating on the unit circle, i.e. for $z = e^{j\Omega}$

$$H(e^{j\Omega}) = \frac{b_0 + b_1 e^{-j\Omega} + \ldots + b_L e^{-jL\Omega}}{1 + a_1 e^{-j\Omega} + \ldots + a_M e^{-jM\Omega}}$$

The Frequency Response Function (Cont.)

The frequency response function and is a complex and periodic
With period *2o*. The normalized frequencies Ω are associated to the sampling frequencies f_s by

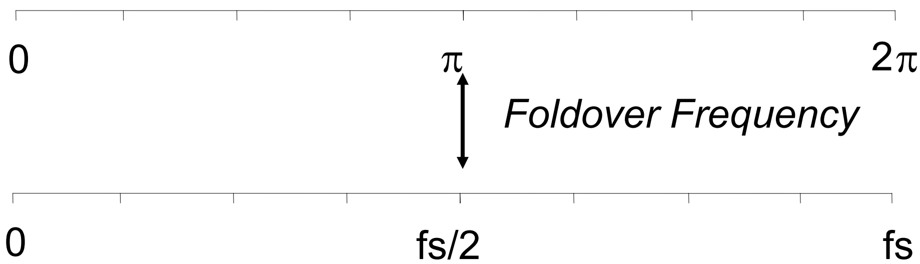
Sampling period

$$\Omega = \omega\, T = 2\pi \; \frac{f}{f_s}$$

rad *rad/s* *Sampling frequency*

where f_s is the sampling frequency and f is any frequency of interest. In practice, one detemines the frequency response up to half the sampling frequency (fold-over frequency).

The Frequency Response Function (Cont.)



•The frequency response is usually plotted w.r.t. normalized frequencies (Ω)

•The frequency response is periodic with period f_s **(2 π)**

•Since frequencies of interest are up to the bandwidth of the analog signal the spectrum is usually plotted up to **f_s/2, (π)** ***the foldover frequency***

The Frequency Response and Poles and Zeros

The magnitude frequency response function

$$\left|H\left(e^{j\Omega}\right)\right| = G\,\frac{\prod_{i=1}^{L}\left|e^{j\Omega}-\zeta_i\right|}{\prod_{i=1}^{M}\left|e^{j\Omega}-p_i\right|}$$

The phase frequency response function

$$\arg\left(H\left(e^{j\Omega}\right)\right)=\sum_{i=1}^{L}\arg\left(e^{j\Omega}-\zeta_i\right)-\sum_{i=1}^{M}\arg\left(e^{j\Omega}-p_i\right)$$

Remarks on Effects of Poles and Zeros on $H(e^{j\Omega})$

•Poles tend to create peaks in the magnitude frequency response

•Zeros tend to create valleys in the magnitude frequency response

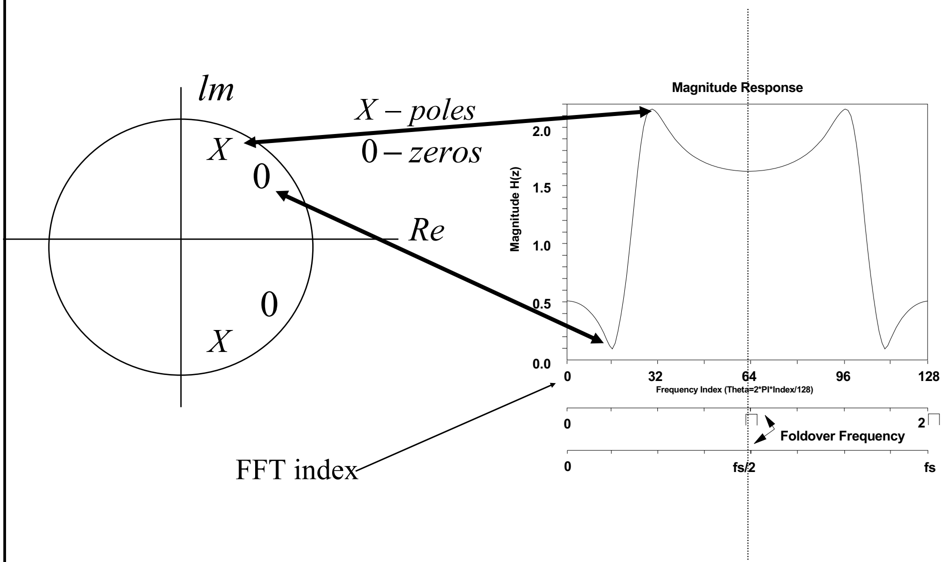
•Selective filters are designed efficiently by placing poles close to the unit circle

•Sharp notches are achieved efficiently with zeros placed very close to the unit circle

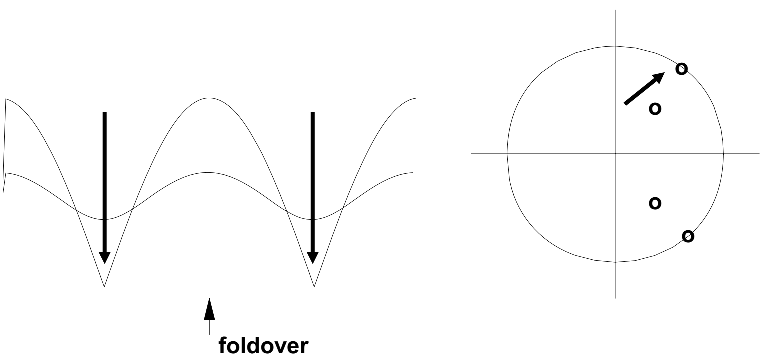
•if restricted to an all-pole filter a sharp notch in the frequency response will require high order design (many poles).

•if restricted to an all-zero (FIR) filter a sharp (high Q) peak in the frequency response will require many zeros (long impulse response or high order FIR).

Z plane and Frequency Magnitude Response

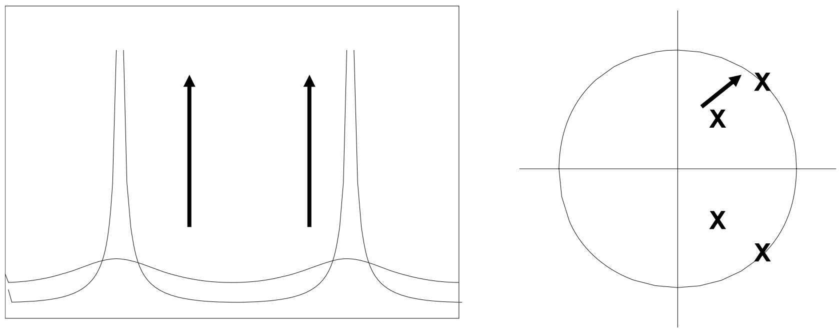


Zero Locations and Frequency Response



Moving zeros towards the unit circle creates sharper valleys

Pole Locations and Frequency Responses



As the poles move towards the unit circle we get sharper peaks and if the poles are placed on the unit circle we get an oscillator.

Computing Filter Responses Using the Inverse Z-Transform

Partial Fractions: Given a the transfer function

$$H(z) = \frac{b_0 z^L + b_1 z^{L-1} + \ldots + b_L}{z^M + a_1 z^{M-l} + \ldots + a_M}$$

for distinct poles write H(z) as:

$$H(z) = c_0 + c_1 \frac{z}{z - p_1} + \ldots + c_M \frac{z}{z - p_M}$$

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Inverse Z-Transform

The coefficients c_j are found using partial fractions, and given the transform pairs:

$$\left\{ p_i^n , n \geq 0 \right\} \leftrightarrow \left\{ \frac{z}{z - p_i} , |z| > |p_i| \right\}$$

$$\left\{ - p_i^n , n < 0 \right\} \leftrightarrow \left\{ \frac{z}{z - p_i} , |z| < |p_i| \right\}$$

as well as the ROC, one can find the sequence corresponding to H(z).

Partial Fractions

Given a rational function:

$$H(z) = \frac{b_0 z^L + b_1 z^{L-1} + \ldots + b_L}{z^M + a_1 z^{M-1} + \ldots + a_M}$$

If M>=L for distinct poles we can write H(z) as:

$$H(z) = c_0 + c_1 \frac{z}{z - p_1} + \ldots + c_M \frac{z}{z - p_M}$$

where c_i are constants to be determined

Partial Fractions Example

Given the second order function

$$H(z) = \frac{z^2 + z}{z^2 - \frac{5}{6}z + \frac{1}{6}}$$

Use partial fractions to write ***H(z)*** as :

$$H(z) = 9\frac{z}{z - \frac{1}{2}} - 8\frac{z}{z - \frac{1}{3}}$$

Partial Fractions Example (Cont.)

and given the ROC

$$|z| > \frac{1}{2}$$

then

$$h(n) = 9\left(\frac{1}{2}\right)^n - 8\left(\frac{1}{3}\right)^n, \; n \geq 0$$

If the ROC is defined differently then the time domain sequences will be different.

Partial Fractions Example (Cont.)

For example, ROC: $\frac{1}{3} < |z| < \frac{1}{2}$

yields
$$h(n) = -8\left(\frac{1}{3}\right)^n, \; n \geq 0$$

$$h(n) = -9\left(\frac{1}{2}\right)^n, \; n < 0$$

Partial Fractions Example (Cont.)

For the ROC: $|z| < \frac{1}{3}$

yields

$$h(n) = 8\left(\frac{1}{3}\right)^n - 9\left(\frac{1}{2}\right)^n, n < 0$$

Inverse Z-Transform - Repeated Poles

Given a rational function: $H(z) = \frac{z^2}{\left(z - \frac{1}{6}\right)^2}$

then the time domain sequence is For ROC:

$|z| > \frac{1}{6}$ $\quad h(n) = (n+1)\left(\frac{1}{6}\right)^n, n \geq 0$

$|z| < \frac{1}{6}$ $\quad h(n) = -(n+1)\left(\frac{1}{6}\right)^n, n < 0$

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Inverse Z-Transform using the Residue Theorem

We will restrict our discussion to causal sequences. Extension to non-causal is straight-forward. Given the z-transform:

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

Cauchy's Integral Theorem:

$$\oint_c X(z) z^{k-1} dz = \oint_c \sum_{n=0}^{\infty} x(n) z^{k-n-1} dz$$

Inverse Z-Transform using the Residue Theorem (Cont.)

If the integration is counterclockwise on a contour which is within the ROC and includes the unit circle, then

$$\oint_c X(z) z^{k-1} dz = \sum_{n=0}^{\infty} x(n) \oint_c z^{k-n-1} dz$$

If the path encloses the origin then

$$\oint_c z^{k-n-1} dz = 2\pi j \delta(k-n)$$

therefore $$x(n) = \frac{1}{2\pi j} \oint_c X(z) z^{n-1} dz$$

Note the similarity with the inverse DTFT which is a special case of the z transform

Inverse Z-Transform using the Residue Theorem (Cont.)

Cauchy's residue theorem states that for rational polynomials X(z) the integral above can be computed as a sum of residues, that is given:

$$X(z) = \frac{A(z)}{(z-p_1)(z-p_2)\ldots(z-p_M)}$$

then

$$x(n) = \sum_{i=1}^{M} res \quad [z^{n-1} X(z)]_{z=p_i}$$

Inverse Z-Transform using the Residue Theorem (Cont.)

For poles of multiplicity m

$$res[z^{n-1}X(z)]_{z=p_i} = \frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}}\left\{(z-p_i)^m z^{n-1}X(z)\right\}\right]_{z=p_i}$$

For single pole

$$res[z^{n-1}X(z)]_{z=p_i} = [(z-p_i)z^{n-1}X(z)]_{z=p_i}$$

<u>Examples:</u> Single pole

$$X(z) = \frac{z}{z-a}$$

Inverse Z-Transform using the Residue Theorem (Cont.)

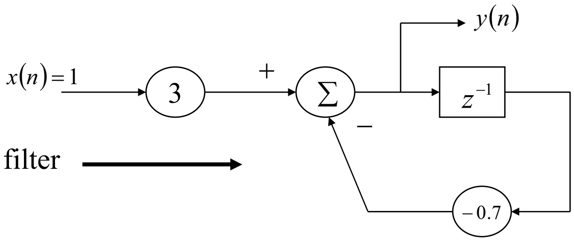
$$res[z^{n-1}X(z)]_{z=a} = \left[(z-a)z^{n-1}\frac{z}{z-a}\right]_{z=a} = a^n$$

Pole of multiplicity 2

$$X(z) = \frac{z^2}{(z-a)^2}$$

$$res[z^{n-1}X(z)]_{z=a} = \left[\frac{d}{dz}\left\{(z-a)^2 z^{n-1}\frac{z^2}{(z-a)^2}\right\}\right]_{z=a} = (n+1)a^n$$

Example: Steady-State and Transient Response of Digital Filters Using the inverse z-transform



Consider the first order IIR filter

$$H(z) = \frac{3z}{(z-0.7)} \qquad X(z) = \frac{z}{(z-1)}$$

$$Y(z) = H(z)\,X(z) = \frac{3z^2}{(z-1)(z-0.7)}$$

Example: Steady-State and Transient Response (Cont.)

Steady-state pole of i/p

transient pole of filter

$$y(n) = res[z^{n-1}Y(z)]_{z=1} + res[z^{n-1}Y(z)]_{z=0.7}$$

$$y(n) = 3\left[\frac{(z-1)z^{n-1}z^2}{(z-0.7)(z-1)}\right]_{z=1} + 3\left[\frac{(z-0.7)z^{n-1}z^2}{(z-1)(z-0.7)}\right]_{z=0.7}$$

$$y(n) = 10 - 7(0.7)^n, n \geq 0$$

Example:Steady-State and Transient Response

$$y(n) = 10 - 7(0.7)^n, n \geq 0$$