Digital Signal Processing

Lectures 4 & 5

Discrete-time Linear Systems
Digital Filters

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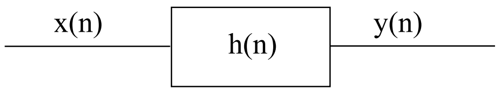
Discrete-time Linear Systems – Digital Filters



The output is produced by convolving the input with the impulse response

$$y(n) = \sum_{m=-\infty}^{\infty} h(m)x(n-m) = h(m) * x(m)$$

This operation can also involve a finite-length impulse response(FIR) sequence

$$y(n) = \sum_{m=0}^{L} h(m)x(n-m)$$

An FIR filter is programmed using a multiply-accumulate instruction

Some Definitions

•A digital filter is linear if it has the property of generalized superposition

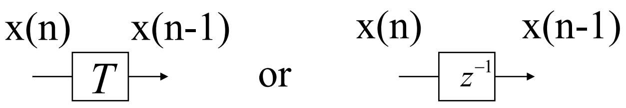
•A digital filter is causal if it non anticipatory, i.e., the present output does not depend on future inputs.

•All real-time systems are causal.

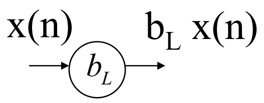
• Non-causalities arise in image processing where the signal indexes are spatial instead of temporal.

• Unless otherwise stated all systems in this course will be assumed causal

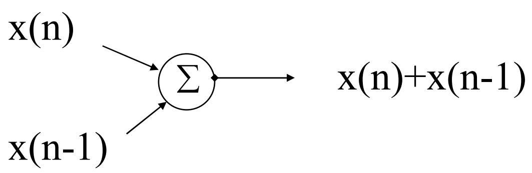
Some More Definitions



;unit delay

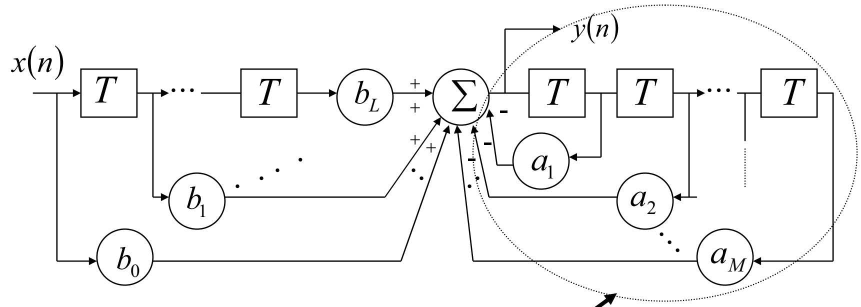


;signal scaling by a filter coefficient



;signal addition

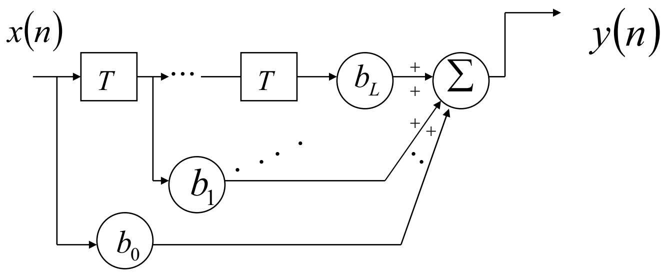
IIR Digital Filter Structure



feedback

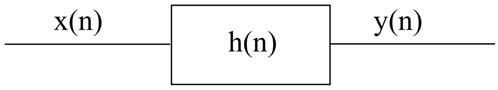
$$y(n) = \sum_{i=0}^{L} b_i x(n-i) - \sum_{i=1}^{M} a_i y(n-i)$$

FIR Digital Filter Structure



$$y(n) = \sum_{i=0}^{L} b_i x(n-i)$$

Two i/p-o/p Equations for Digital Filters



One can compute the output using the *convolution sum*

$$y(n) = \sum_{m=-\infty}^{\infty} h(m)x(n-m) = \sum_{m=-\infty}^{\infty} x(m)h(n-m)$$

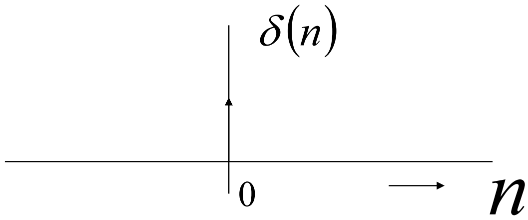
or by using the *difference equation*

$$y(n) = \sum_{i=0}^{L} b_i x(n-i) - \sum_{i=1}^{M} a_i y(n-i)$$

Remark: The impulse response *h(n)* can be determined by solving the difference equation.

Unit Impulse

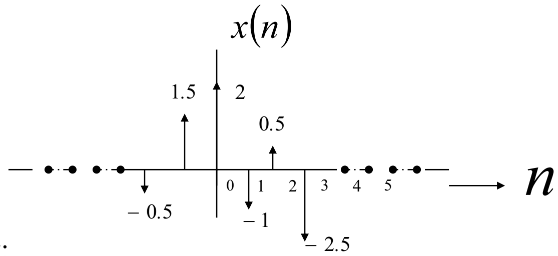
The analysis of digital filters in the frequency domain is facilitated using sinusoids. In the time domain a unique input signal is used for analysis, namely the unit impulse. That is defined as:



$$\delta(n) = \begin{cases} 1 & \text{for} \quad n=0 \\ 0 & \text{elsewhere} \end{cases}$$

Signal Representation with Unit Impulses

Any discrete-time signal may be represented by a linear combination of unit impulses



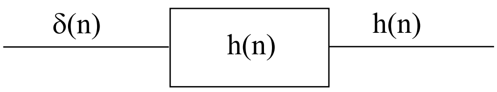
is represented by:

$$x(n) = -.5\delta(n+2) + 1.5\delta(n+1) + 2\delta(n)$$
$$-\,\delta(n-1) + .5\delta(n-2) - 2.5\delta(n-3)$$

Impulse Response

The response of a digital filter to a unit impulse is known as impulse response and is given by

$$h(n) = b_0\delta(n) + b_1\delta(n-1) + \ldots + b_L\delta(n-L) -$$
$$a_1 h(n-1) - a_2 h(n-2) - \ldots - a_M h(n-M)$$



Finite-Length Impulse Response (FIR)

If the digital filter has no feedback terms the impulse response is finite length

$$h(n) = b_0\delta(n) + b_1\delta(n-1) + \ldots + b_L\delta(n-L)$$

Note that

$$h(0) = b_0$$

$$h(1) = b_1$$

$$h(2) = b_2$$

$$\vdots$$

$$h(L) = b_L$$

Remark: The filter has a finite-length impulse response and is called FIR. The values of the impulse response sequence are the coefficients themselves. The filter is always stable.

Example – The Moving Average Filter

$$y(n) = \frac{1}{L+1}\sum_{i=0}^{L} x(n-i)$$

$$h(n) = \frac{1}{L+1}\{\delta(n) + \delta(n-1) + \ldots + \delta(n-L)\}$$

$$h(n) = \frac{1}{L+1} \qquad 0 \le n \le L$$

Remark: The moving average is essentially a low-pass (smoothing) filter. Later on we will see that this filter is also optimal in estimation problems.

Infinite-Length Impulse Response (IIR)

If the digital filter has feedback terms then the impulse response is infinite length

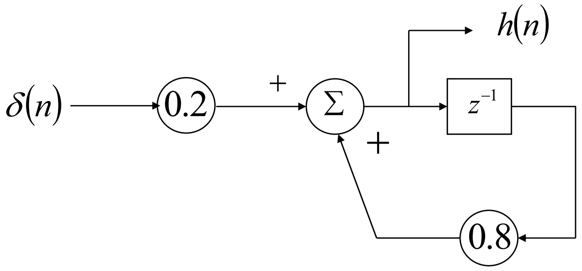
$$h(n) = \sum_{i=0}^{L} b_i \delta(n-i) - \sum_{i=1}^{M} a_i h(n-i)$$

Example: $$h(n) = \delta(n) - a_1 h(n-1)$$

$$h(0) = 1 \quad h(1) = -a_1 \quad h(2) = a_1^2 \qquad :: \qquad h(n) = (-a_1)^n$$

Remark: Note that is the coefficient $\boldsymbol{a_1}$ has magnitude larger than one the the impulse response will go to infinity and hence the filter would be unstable.

IIR – Another First Order Example



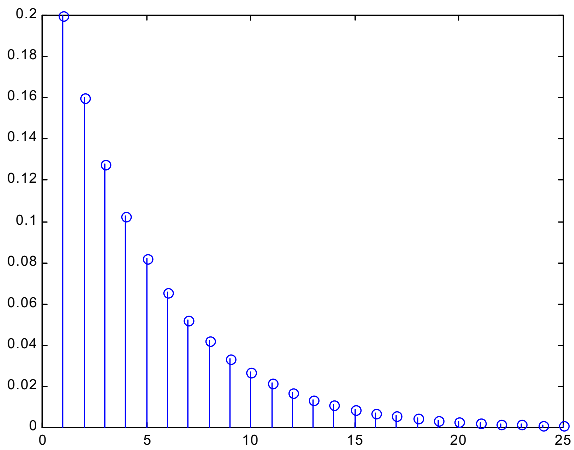
$$h(n) = 0.2\delta(n) + 0.8h(n-1)$$

$$h(n) = 0.2\ (0.8)^n \qquad n \geq 0$$

Remark: This particular IIR filter is also a low-pass filter behaving in similar manner like the the averaging FIR filter.

IIR – Another First Order Example (Plot)

$$h(n) = 0.2\ (0.8)^n \qquad n \geq 0$$



Impulse Response of M-th Order IIR Filters

We have noticed that for a first order causal IIR filter

$$y(n) = x(n) - a_1 y(n-1) \quad \textbf{is} \quad h(n) = (-a_1)^n \qquad n \geq 0$$

For an M-th order causal IIR filter the impulse response

$$y(n) = \sum_{i=0}^{L} b_i x(n-i) - \sum_{i=1}^{M} a_i y(n-i)$$

the impulse response, assuming distinct roots, is

$$h(n) = c_1 p_1^n + c_2 p_2^n + \ldots + c_M p_M^n \qquad n \geq 0$$

Remark: c_1, c_2, .. are constants. p_1., p_2,… are the poles of the filter. Note that if all the poles have magnitude less than one then the filter is stable.

Impulse Response and Causality

A digital filter is causal if the output depends on present and past samples, as opposed to dependence on future samples.

$$y(n) = \sum_{i=0}^{L} b_i x(n-i) - \sum_{i=1}^{M} a_i y(n-i)$$

A causal digital filter is described by a causal impulse response, i.e.,

$$h(n) = 0 \qquad \textbf{\textit{for}} \qquad \boldsymbol{n < 0}$$

therefore

$$y(n) = \sum_{m=0}^{\infty} h(m) x(n-m)$$

Impulse Response and Stability

For the causal digital filter

$$y(n) = \sum_{m=0}^{\infty} h(m) x(n-m)$$

Bounded Input Bounded Output (BIBO) stability is defined as

$$\sum_{k=0}^{\infty} \left| h(k) \right| < \infty$$

The condition above is guaranteed if

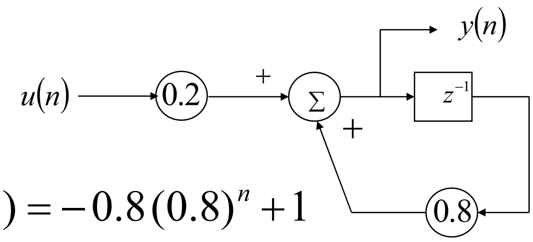
$$\left| p_i \right| < 1 \quad \textbf{\textit{for all}} \quad \boldsymbol{i = 1, 2, \ldots, M}$$

Transient and Steady State Response of Digital Filters

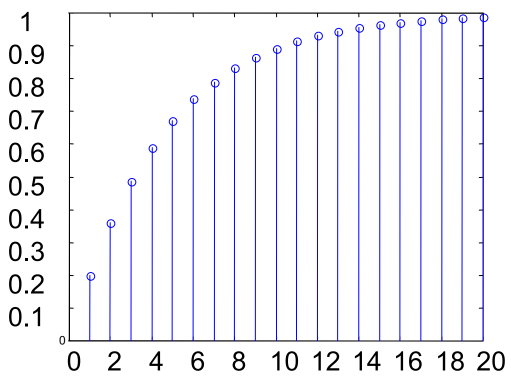
- Digital Filters have memory and hence they have transient and a steady-state response
- An FIR filter has L memory elements keeping past data. Therefore it will take L samples to reach steady state
- For an IIR filter the transient state is determined by solving the difference equation.

Example of Transient and Steady State Response

$$u(n) = 1 \qquad n \geq 0$$



$$y(n) = y^{tr}(n) + y^{ss}(n) = -0.8(0.8)^n + 1$$



Transient Response of IIR Filters

The transient response for an IIR filter is of the form:

$$y^{tr}(n) = c_1 p_1^n + c_2 p_2^n + \ldots + c_M p_M^n$$

c_i , $i = 1, 2, \ldots, m$ are constants evaluated using initial conditions

p_i , $i = 1, 2, \ldots, m$ are the poles of the filter which are determined by solving the equation below

$$p^M + a_1 p^{M-1} + a_2 p^{M-2} + \ldots + a_{M-1} p + a_M = 0$$

Steady-State Sinusoidal Response of Digital Filters

A special case of interest is the steady-state response to input sinusoids and is formulated as follows. For the IIR filter

$$y(n) = \sum_{i=0}^{L} b_i x(n-i) - \sum_{i=1}^{M} a_i y(n-i)$$

The frequency response function is given by:

$$H(e^{j\Omega}) = \frac{b_0 + b_1 e^{-j\Omega} + b_2 e^{-j2\Omega} + \ldots + b_L e^{-jL\Omega}}{1 + a_1 e^{-j\Omega} + a_2 e^{-j2\Omega} + \ldots + a_M e^{-jM\Omega}}$$

Steady-State Sinusoidal Response of Linear Discrete Systems (Cont.)

The frequency response function is periodic and an example of the steady state sinusoidal response is given below

if:

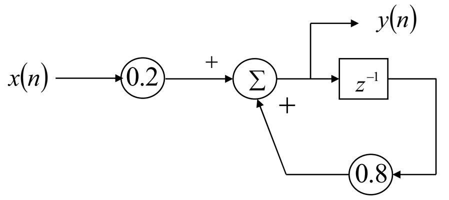
$$x(n) = \sin(n\Omega)$$

then:

$$y^{ss}(n) = \left|H(e^{j\Omega})\right| \sin(\Omega n + \angle H(e^{j\Omega}))$$

$$\Omega = \frac{2\pi f}{f_s} \qquad \text{;normalized frequency}$$

Example of Steady State Sinusoidal Response



$$H(e^{j\Omega}) = \frac{0.2}{1-0.8e^{-j\Omega}}$$

The filter is excited by a 500 Hz sinusoid and the Sampling rate is 2000Hz.

$$x(n) = \sin(2\pi n 500/2000) = \sin(\frac{\pi n}{2})$$

$$y^{ss}(n) = \left|\frac{0.2}{1+0.8j}\right| \sin(\frac{\pi n}{2} + \arg(\frac{0.2}{1+0.8j}))$$

Frequency Response Plot

$$H(e^{j\Omega}) = \frac{0.2}{1 - 0.8e^{-j\Omega}}$$