Digital Signal Processing

Lecture 7

The Discrete-time Fourier Transform (DTFT)

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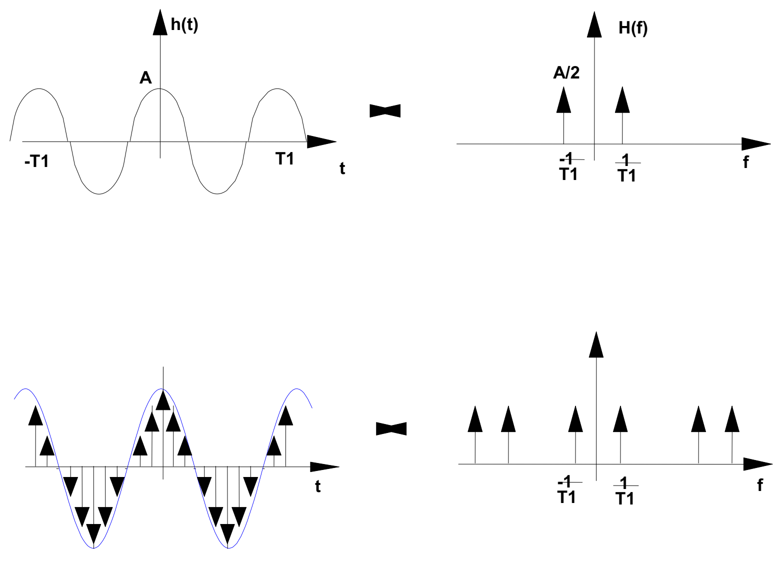
From the Continuous Fourier to the Discrete-time Fourier Transform



From the Continuous Fourier to the Discrete-time Fourier Transform

The frequency domain representation of continuous signals is given by

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

If we consider a sampled signal $x_s(t)$, that is

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

then its F.T. is

$$X_s(\omega) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) e^{-j\omega t} dt$$

$$X_s(\omega) = \sum_{n=-\infty}^{\infty} x(nT) e^{-j\omega nT}$$

The Discrete-Time Fourier Transform

The F.T can be also written as

$$X_s(\omega) = \sum_{n=-\infty}^{\infty} x(nT) e^{-j\omega nT}$$

Note that $$X_s(\omega + \omega_s) = X_s(\omega)$$

By defining $$\Omega = \omega T = 2\pi fT = 2\pi \frac{f}{f_s}$$

and omitting the symbol T and the subscript s one can write

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\Omega}$$

which is known as the <u>Discrete-time Fourier Transform</u> of *x(n)*.

The Inverse Discrete-time Fourier Transform

The Inverse DTFT is

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{jn\Omega} d\Omega$$

A DTFT pair is denoted as

$$x(n) \leftrightarrow X(e^{j\Omega})$$

Unlike the CFT the DTFT is a periodic complex function with period 2π. The DTFT is a linear transformation and has properties similar to those of F.T.

Relation to the Z-transform

For $z = e^{j\Omega}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \longrightarrow X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\Omega}$$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} \longrightarrow H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h(n) e^{-jn\Omega}$$

Selected Properties of the DTFT

Linearity: $\alpha x(n) + \beta y(n) \leftrightarrow \alpha X(e^{j\Omega}) + \beta Y(e^{j\Omega})$

Shifting: $x(n \pm m) \leftrightarrow e^{\pm j\Omega m} X(e^{j\Omega})$

Time Convolution: $\{x(n)\} * \{y(n)\} \leftrightarrow X(e^{j\Omega})Y(e^{j\Omega})$

Freq. Convolution: $\{x(n)\}\{y(n)\} \leftrightarrow \frac{1}{2\pi} X(e^{j\Omega}) * Y(e^{j\Omega})$

Parseval's Theorem: $\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\Omega})|^2 d\Omega$

The DTFT of a finite sequence

if::

$$x(n) = \begin{cases} 1\ldots\ldots, 0 \le n \le N-1 \\ 0\ldots\ldots, elsewhere \end{cases}$$

then

$$X(e^{j\Omega}) = \sum_{n=0}^{N-1} e^{-jn\Omega} = \frac{1 - e^{-jN\Omega}}{1 - e^{-j\Omega}}$$

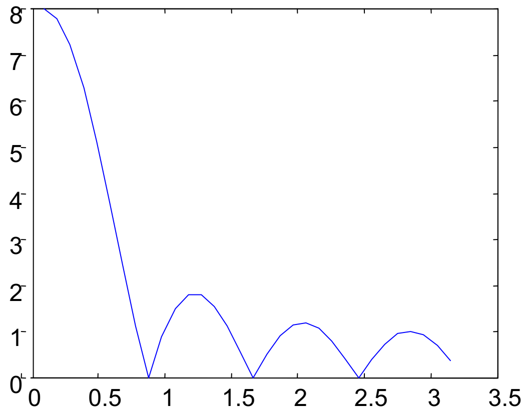
or

$$X(e^{j\Omega}) = e^{-j(N-1)\Omega/2} \frac{\sin(N\Omega/2)}{\sin(\Omega/2)}$$

Remark: The sin(.)/sin(.) function is known as a *digital sinc* or a *Dirichlet* function.

The DTFT of a finite sequence (Cont.)

$$x(n) = \begin{cases} 1 \ldots\ldots, 0 \le n \le 7 \\ 0 \ldots\ldots, elsewhere \end{cases}$$



The DTFT of a finite sequence (Cont.)

$$x(n) = \begin{cases} 1 \ldots\ldots, 0 \le n \le 15 \\ 0 \ldots\ldots, elsewhere \end{cases}$$