The z-Transform in DSP
Lecture 8 & 9

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The Z-Transform

•The z-transform plays a similar role in DSP as the Laplace transform in analog circuits and systems.

•It provides intuition that is sometimes not evident in time-domain analysis

•Simplifies time-domain operations – time domain-convolution maps to Z-domain multiplication

•Used to define transfer functions

•Could be used to determine responses of systems using a table look-up process

The Z-Transform - Definition

Given the sequence: $x(n)$

its Z-transform is $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

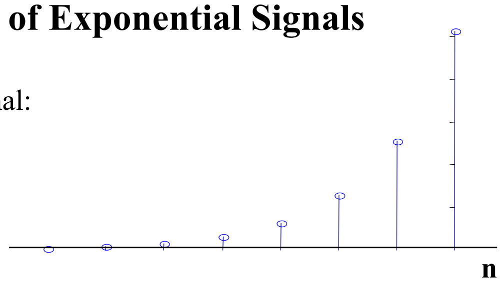
For causal sequences, i.e., $x(n) = 0 \quad for \quad n < 0$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

Z transform of Exponential Signals

Find the Z-transform of the signal:

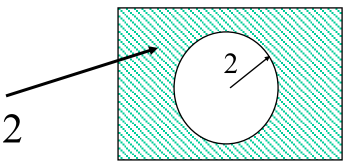
$$
\begin{aligned}
x(n) &= 2^n \quad \text{for} \quad n \geq 0 \\
&= 0 \quad \text{for} \quad n < 0
\end{aligned}
$$



$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} 2^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n$$

The above sum converges to

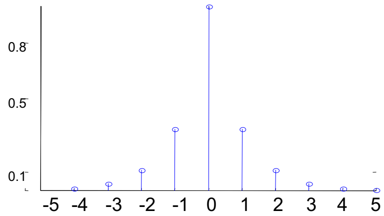
$$X(z) \rightarrow \frac{z}{z-2} \qquad \text{if} \qquad |z| > 2$$



Double Sided Exponential

Find X(z) if the signal is

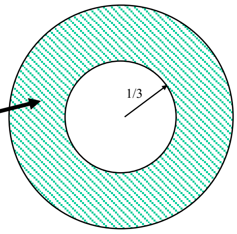
$$x(n) = (1/3)^n \quad \text{for} \quad n \geq 0$$
$$= 3^n \quad \text{for} \quad n < 0$$



$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^{|n|} z^{-n} = \sum_{n=1}^{\infty} \left(\frac{z}{3}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{3z}\right)^n$$

$$X(z) = \frac{z/3}{1-(z/3)} + \frac{1}{1-(1/3z)}$$

if $\quad 1/3 < |z| < 3$



Selected Properties of the Z-Transform

Linearity: if: $x(n) \leftrightarrow X(z)$ and $y(n) \leftrightarrow Y(z)$

then

$$\alpha x(n) + \beta y(n) \leftrightarrow \alpha X(z) + \beta Y(z)$$

Shifting: $$x(n \pm m) \leftrightarrow z^{\pm m} X(z)$$

Convolution: $$x(n) * y(n) \leftrightarrow X(z)Y(z)$$

Scaling: $$a^n x(n) \leftrightarrow X(z/a)$$

Selected Z-Transform Pairs

Unit-Impulse: $\delta(n) \leftrightarrow 1$

Sinusoids: $\{\sin(\Omega n), n \geq 0\} \leftrightarrow \left\{ \dfrac{z^{-1}\sin(\Omega)}{1 - 2z^{-1}\cos(\Omega) + z^{-2}}, |z| > 1 \right\}$

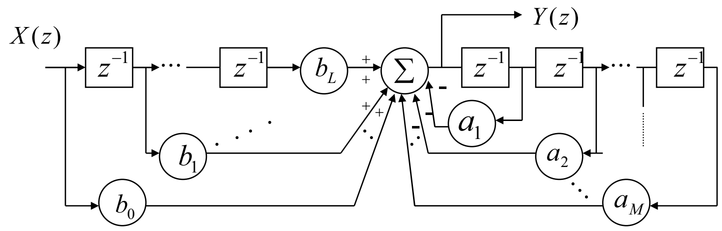
$$\{\cos(\Omega n), n \geq 0\} \leftrightarrow \left\{ \frac{1 - z^{-1}\cos(\Omega)}{1 - 2z^{-1}\cos(\Omega) + z^{-2}}, |z| > 1 \right\}$$

Sampled Unit-Step: $\{1, n \geq 0\} \leftrightarrow \left\{ \dfrac{1}{1 - z^{-1}}, |z| > 1 \right\}$

Exponential Signals: $\{a^n, n \geq 0\} \leftrightarrow \left\{ \dfrac{z}{z - a}, |z| > |a| \right\}$

LECTURE 9

The Transfer Function



$$y(n)=\sum_{i=0}^{L} b_i x(n-i)-\sum_{i=1}^{M} a_i y(n-i)$$

To write the transfer function put the difference equation in the z-domain

$$x(n)\leftrightarrow X(z) \qquad \text{and} \qquad y(n)\leftrightarrow Y(z)$$

$$Y(z)=\sum_{i=0}^{L} b_i\, X(z)\, z^{-i}-\sum_{i=1}^{M} a_i\, Y(z)\, z^{-i}$$

The Transfer Function (Cont.)

$$X(z)\ (\sum_{i=0}^{L} b_i\ z^{-i})=Y(z)\,(1+\sum_{i=1}^{M} a_i\ z^{-i})$$

The transfer function *H(z)* is defined as:

$$H(z)=\frac{Y(z)}{X(z)}=\frac{b_0+b_1 z^{-1}+...+b_L z^{-L}}{1+a_1 z^{-1}+...+a_M z^{-M}}$$

$$H(z)=\frac{\sum_{i=0}^{L} b_i\ z^{-i}}{1+\sum_{i=1}^{M} a_i\ z^{-i}}$$

Note that feedback terms are in the denominator

The Transfer Function and the Impulse Response

Convolution maps to multiplications in the z domain

$$y(n) = x(n) * h(n) \quad \leftrightarrow \quad Y(z) = X(z)H(z)$$

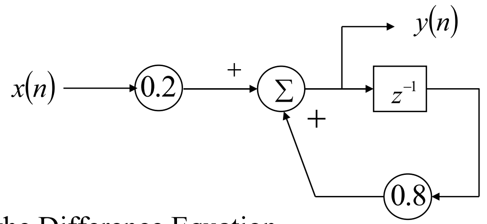
It can be shown that a transfer function *H(z)* is related to the impulse response sequence *h(n)* by:

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

or

$$h(n) \leftrightarrow H(z)$$

Example: Write the Transfer Function of a First Order IIR Filter



Step 1: Write the Difference Equation

$$y(n) = 0.2x(n) + 0.8y(n-1)$$

Step 2: Transform all signals to the z domain

$$y(n) \Leftrightarrow Y(z) \qquad x(n) \Leftrightarrow X(z)$$

$$y(n-1) \Leftrightarrow z^{-1}Y(z)$$

Example: Transfer Function of a First Order IIR Filter (cont.)

Step 3: Since z-transform is a linear operation I can write

$$Y(z) = 0.2\, X(z) + 0.8\, z^{-1}\, Y(z)$$

Step 4: Form the ratio Y(z)/X(z) to get the transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.2}{1-0.8z^{-1}} = \frac{0.2z}{z-0.8}$$

Note below that the impulse response can be written by inspection of H(z)

$$h(n) = 0.2 \quad 0.8^n\, u(n) \;\Leftrightarrow\; H(z) = \frac{0.2z}{z-0.8} \quad |z| > 0.8$$

The Transfer Function of FIR Systems

For the FIR filter the transfer function is:

$$H(z) = b_0 + b_1 z^{-1} + \ldots + b_L z^{-L} = \sum_{i=0}^{L} b_i \; z^{-i}$$

this is also in agreement with

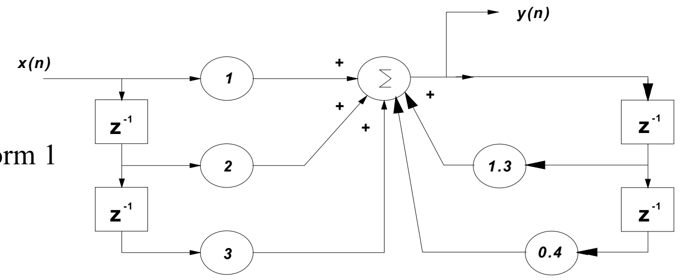
$$H\ (z) = \sum_{n=0}^{L} h(n)\, z^{-n}$$

Note that for FIR filters

$$\{h(0),\ h(1),\ldots,\ \ h(L)\} = \{b_0, b_1, \ldots.,\ \ b_L\}$$

EQUIVALENT FILTER REALIZATIONS

Direct Form 1



Direct Form 2