University of Cyprus
Biomedical Imaging & Applied Optics

Διάλεξη 8

Μετασχηματισμός z

Κεφ. 3

Μετασχηματισμός z

- **Το αντίστοιχο του μετασχηματισμού Laplace στο διακριτό χρόνο**
  - Ανάλυση γραμμικών συστημάτων διακριτού χρόνου
- **ΜΦΔΤ**

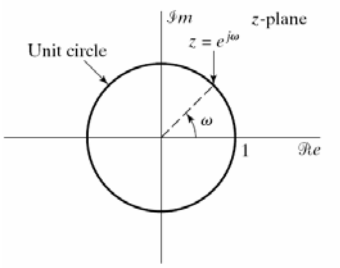
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- **Μετασχηματισμός z**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \qquad z = re^{j\omega}$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}$$

  - Σχηματικά στο πεδίο z (z plane)
  - Για |z|=1 → ΜΦΔΤ
  - ΜΦΔΤ → Μz στο μοναδιαίο κύκλο



•Πού είναι τα ω=0, π, -π;
•Τι σας λέει το σχήμα για την περιοδικότητα του ΜΦΔΤ;

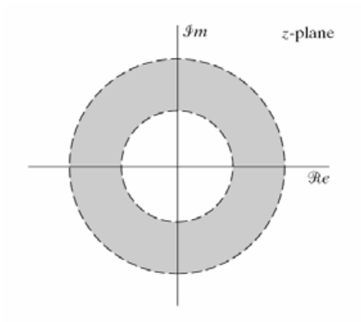
Μετασχηματισμός z

- **Περιοχή Σύγκλισης (Region of Convergence – ROC)**
  - Για να μπορεί να υπολογιστεί ο Μz πρέπει

$$\sum_{n=-\infty}^{\infty} |x[n]|\,|z|^{-n} < \infty \qquad z = re^{j\omega}$$

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$

  - Περιοχή Σύγκλισης (Region of Convergence – ROC)
    - Όλα τα z για τα οποία ισχύει το πιο πάνω
    - Εξαρτάται από το r → Ομόκεντροι κύκλοι
    - Μπορεί να περιλαμβάνει και την αρχή (origin) και το ∞
    - Αν περιλαμβάνει και το r=1 τότε υπάρχει και ο ΜΦΔΧ
  - Λόγω του r μπορεί να υπάρχει σύγκλιση και για συναρτήσεις για τις οποίες δεν συγκλίνει ο ΜΦΔΧ



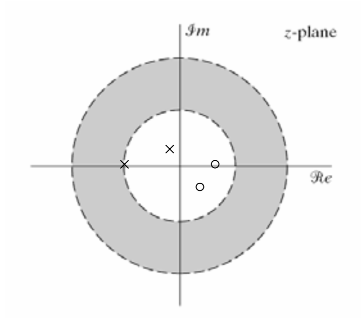
Σημείωση:
Ο Μz είναι μια δυναμοσειρά Laurent και άρα είναι συνεχής για όλες τις τιμές μέσα στη ROC
Τι γίνεται με το ΜΦΔΧ ημιτόνου ή συνημίτονου;

Μετασχηματισμός z

- **Μz ως ρητή συνάρτηση (rational function)**

$$X(z) = \frac{P(z)}{Q(z)}$$

  - Μηδενικά (zeros) o
    - X(z)=0
    - Ρίζες του P(z)
  - Πόλοι (poles) x
    - X(z)= ∞
    - Ρίζες του Q(z) ή/και z=0, z=∞
    - Θα δούμε πως καθορίζουν την ROC



Μετασχηματισμός z

- Παραδείγματα

Example 3.1 Right-Sided Exponential Sequence

Consider the signal $x[n] = a^n u[n]$. Because it is nonzero only for $n \geq 0$, this is an example of a *right-sided* sequence. From Eq. (3.2),

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$$

For convergence of $X(z)$, we require that

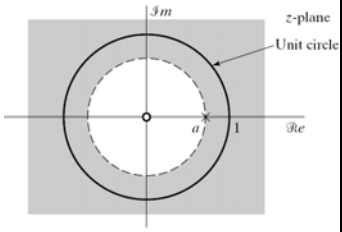
$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty.$$

Thus, the region of convergence is the range of values of z for which $|az^{-1}| < 1$ or, equivalently, $|z| > |a|$. Inside the region of convergence, the infinite series converges to

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad |z| > |a|. \tag{3.10}$$

Here we have used the familiar formula for the sum of terms of a geometric series. The z-transform has a region of convergence for any finite value of $|a|$. The Fourier transform of $x[n]$, on the other hand, converges only if $|a| < 1$. For $a = 1$, $x[n]$ is the unit step sequence with z-transform

$$X(z) = \frac{1}{1 - z^{-1}}, \qquad |z| > 1. \tag{3.11}$$



Μετασχηματισμός z

- Παραδείγματα

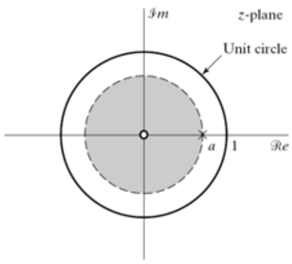
Example 3.2 Left-Sided Exponential Sequence

Now let $x[n] = -a^n u[-n-1]$. Since the sequence is nonzero only for $n \leq -1$, this is a *left-sided* sequence. Then

$$\begin{aligned} X(z) &= -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} \\ &= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n. \end{aligned} \tag{3.12}$$

If $|a^{-1}z| < 1$ or, equivalently, $|z| < |a|$, the sum in Eq. (3.12) converges, and

$$X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad |z| < |a|. \tag{3.13}$$



• Παραδείγματα

Example 3.3 Sum of Two Exponential Sequences

Consider a signal that is the sum of two real exponentials:

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(-\tfrac{1}{3}\right)^n u[n]. \tag{3.14}$$

The z-transform is then

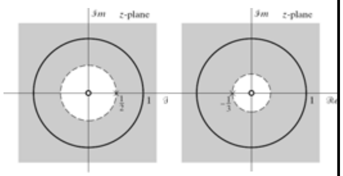
$$X(z) = \sum_{n=-\infty}^{\infty} \left\{ \left(\tfrac{1}{2}\right)^n u[n] + \left(-\tfrac{1}{3}\right)^n u[n] \right\} z^{-n}$$

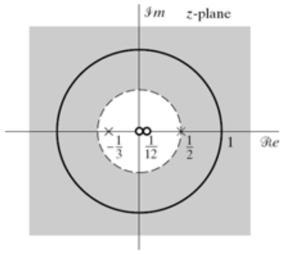
$$= \sum_{n=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} \left(-\tfrac{1}{3}\right)^n u[n] z^{-n} \tag{3.15}$$

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{2}z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\tfrac{1}{3}z^{-1}\right)^n$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} = \frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}$$

$$= \frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}. \tag{3.16}$$





• Παραδείγματα

Example 3.5 Two-Sided Exponential Sequence

Consider the sequence

$$x[n] = \left(-\tfrac{1}{3}\right)^n u[n] - \left(\tfrac{1}{2}\right)^n u[-n-1]. \tag{3.20}$$

Note that this sequence grows exponentially as $n \to -\infty$. Using the general result of Example 3.1 with $a = -\frac{1}{3}$, we obtain

$$\left(-\tfrac{1}{3}\right)^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |z| > \tfrac{1}{3},$$

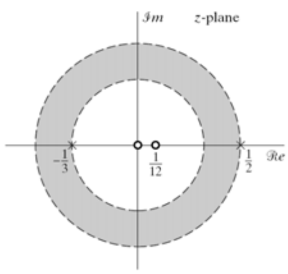
and using the result of Example 3.2 with $a = \frac{1}{2}$ yields

$$-\left(\tfrac{1}{2}\right)^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| < \tfrac{1}{2}.$$

Thus, by the linearity of the z-transform,

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad \tfrac{1}{3} < |z|, \quad |z| < \tfrac{1}{2},$$

$$= \frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}. \tag{3.21}$$



Μετασχηματισμός z

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Μετασχηματισμός z

• **Ιδιότητες της περιοχής σύγκλισης**

PROPERTY 1: The ROC is a ring or disk in the z-plane centered at the origin; i.e., $0 \le r_R < |z| < r_L \le \infty$.

PROPERTY 2: The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the unit circle.

PROPERTY 3: The ROC cannot contain any poles.

PROPERTY 4: If $x[n]$ is a *finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \le n \le N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.

PROPERTY 5: If $x[n]$ is a *right-sided sequence*, i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

PROPERTY 6: If $x[n]$ is a *left-sided sequence*, i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the *innermost* (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

PROPERTY 7: A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with property 3, not containing any poles.

PROPERTY 8: The ROC must be a connected region.

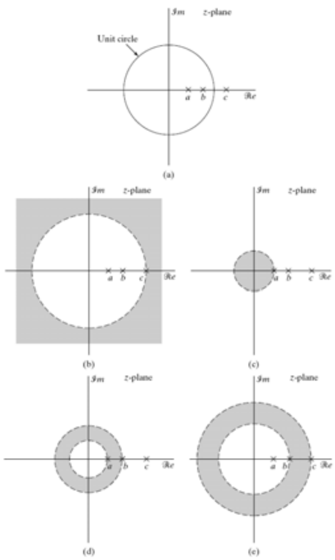


Figure 3.10 Examples of four z-transforms with the same pole-zero locations, illustrating the different possibilities for the region of convergence. Each ROC corresponds to a different sequence: (b) to a right-sided sequence, (c) to a left-sided sequence, (d) to a two-sided sequence, and (e) to a two-sided sequence.

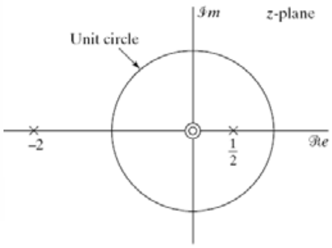
Μετασχηματισμός z

• **Παράδειγμα**

Example 3.7 Stability, Causality, and the ROC

Consider a system with impulse response $h[n]$ for which the z-transform $H(z)$ has the pole–zero plot shown in Figure 3.11. There are three possible ROC's consistent with properties 1–8 that can be associated with this pole-zero plot. However, if we state in addition that the system is stable (or equivalently, that $h[n]$ is absolutely summable and therefore has a Fourier transform), then the ROC must include the unit circle. Thus, stability of the system and properties 1–8 imply that the ROC is the region $\frac{1}{2} < |z| < 2$. Note that as a consequence, $h[n]$ is two sided, and therefore, the system is not causal.

If we state instead that the system is causal, and therefore that $h[n]$ is right sided, then property 5 would require that the ROC be the region $|z| > 2$. Under this condition, the system would not be stable; i.e., for this specific pole–zero plot, there is no ROC that would imply that the system is both stable and causal.



Μετασχηματισμός z

• **Αντίστροφος Μz**

• **Μέθοδοι**
  - Με Επιθεώρηση (By Inspection)
  - Με Ανάπτυξη σε Μερικά Κλάσματα (By Partial Fraction Expansion)
  - Με Ανάπτυξη σε Δυναμοσειρά (By Power Series Expansion)

• **Με Επιθεώρηση (By Inspection)**
  - Με τη βοήθεια του Πίνακα 3.1

$$X(z) = \left(\frac{1}{1 - \frac{1}{2}z^{-1}}\right), \qquad |z| > \tfrac{1}{2}$$

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n]$$

$$X(z) = \left(\frac{1}{1 - \frac{1}{2}z^{-1}}\right), \qquad |z| < \tfrac{1}{2}$$

$$x[n] = -\left(\tfrac{1}{2}\right)^n u[-n-1]$$

Μετασχηματισμός z

- Με Ανάπτυξη σε Μερικά Κλάσματα (By Partial Fraction Expansion)

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

$M < N$

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}} \qquad A_k = (1 - d_k z^{-1}) X(z)\big|_{z=d_k}$$

$M \geq N$

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}} \qquad B_r\text{'s by long division}$$

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1, k\neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m}$$

$$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} [(1 - d_i w)^s X(w^{-1})] \right\}_{w=d_i^{-1}}$$

Μετασχηματισμός z

- Τι σημαίνουν τα κλάσματα

$$B_r z^{-r} \overset{z}{\longleftrightarrow} B_r \delta[n-r]$$

  - Μετατοπισμένες κρουστικές συναρτήσεις

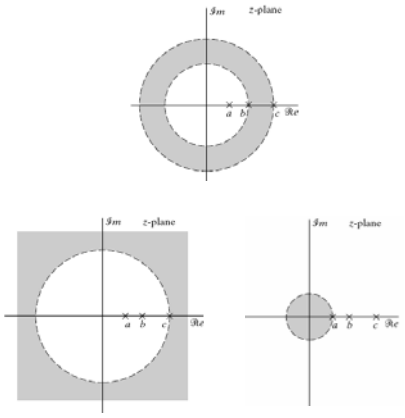
$$\frac{A_k}{1 - d_k z^{-1}} \overset{z}{\longleftrightarrow} (d_k)^n u[n] \text{ or } -(d_k)^n u[-n-1]$$

  - Εκθετικά. Αλλά, τι εκθετικά;

ROC $r_R < |z| < r_L$

$(d_k)^n u[n]$ if $|d_k| < r_R$

$-(d_k)^n u[-n-1]$ if $|d_k| > r_L$



Μετασχηματισμός z

• **Παράδειγμα**

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{(1 + z^{-1})^2}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}, \qquad |z| > 1$$

$M = N = 2$

the poles are all first order

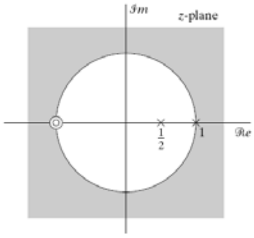
$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

$$\begin{array}{r|l} & \quad 2 \\ \hline \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & z^{-2} + 2z^{-1} + 1 \\ & \underline{z^{-2} - 3z^{-1} + 2} \\ & \quad 5z^{-1} - 1 \end{array}$$

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}$$

$$A_1 = \left[\left(2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}\right)\left(1 - \frac{1}{2}z^{-1}\right)\right]_{z=1/2} = -9$$

$$A_2 = \left[\left(2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}\right)(1 - z^{-1})\right]_{z=1} = 8.$$



$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}$$

$$2 \stackrel{z}{\longleftrightarrow} 2\delta[n],$$

$$\frac{1}{1 - \frac{1}{2}z^{-1}} \stackrel{z}{\longleftrightarrow} \left(\frac{1}{2}\right)^n u[n]$$

$$\frac{1}{1 - z^{-1}} \stackrel{z}{\longleftrightarrow} u[n].$$

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$$

Μετασχηματισμός z

• **Με Ανάπτυξη σε Δυναμοσειρά (By Power Series Expansion)**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
$$= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

$\delta[n-m]$	$\stackrel{z}{\longleftrightarrow}$	z^{-m}	ROC	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)

Μετασχηματισμός z

- **Παράδειγμα**

$$X(z) = z^2\left(1 - \tfrac{1}{2}z^{-1}\right)(1 + z^{-1})(1 - z^{-1})$$

$$X(z) = z^2 - \tfrac{1}{2}z - 1 + \tfrac{1}{2}z^{-1}$$

$$x[n] = \begin{cases} 1, & n = -2, \\ -\frac{1}{2}, & n = -1, \\ -1, & n = 0, \\ \frac{1}{2}, & n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$x[n] = \delta[n+2] - \tfrac{1}{2}\delta[n+1] - \delta[n] + \tfrac{1}{2}\delta[n-1]$$

- **Παράδειγμα**

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|$$

$$\begin{array}{r|l} & 1 + az^{-1} + a^2z^{-2} + \cdots \\ \hline 1 - az^{-1} & 1 \\ & 1 - az^{-1} \\ \hline & \quad az^{-1} \\ & \quad az^{-1} - a^2z^{-2} \\ \hline & \qquad a^2z^{-2} \quad \cdots \end{array}$$

$$\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \cdots$$

$$x[n] = a^n u[n].$$

Μετασχηματισμός z

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Section Reference	Sequence	Transform	ROC
	$x[n]$	$X(z)$	R_x $\quad r_R < \lvert z\rvert < r_L$
	$x_1[n]$	$X_1(z)$	R_{x_1}
	$x_2[n]$	$X_2(z)$	R_{x_2}
3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
3.4.2	$x[n - n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$ $\quad \lvert z_0\rvert r_R < \lvert z\rvert < \lvert z_0\rvert r_L$
3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞
3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
	$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
	$\mathcal{J}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$ $\quad 1/r_L < \lvert z\rvert < 1/r_R$
	$x[n]$ is real $x[-n]$	$X(1/z)$	
3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
3.4.8	Initial-value theorem:		
	$x[n] = 0, \quad n < 0$ $\qquad \lim\limits_{z\to\infty} X(z) = x[0]$		

Μετασχηματισμός z

• Παραδείγματα

Example 3.15 Exponential Multiplication

Starting with the transform pair

$$u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-z^{-1}}, \qquad |z| > 1, \tag{3.60}$$

we can use the exponential multiplication property to determine the z-transform of

$$x[n] = r^n \cos(\omega_0 n)u[n]. \tag{3.61}$$

First, $x[n]$ is expressed as

$$x[n] = \tfrac{1}{2}(re^{j\omega_0})^n u[n] + \tfrac{1}{2}(re^{-j\omega_0})^n u[n].$$

Then, using Eq. (3.60) and the exponential multiplication property, we see that

$$\tfrac{1}{2}(re^{j\omega_0})^n u[n] \overset{z}{\longleftrightarrow} \frac{\frac{1}{2}}{1-re^{j\omega_0}z^{-1}}, \qquad |z| > r,$$

$$\tfrac{1}{2}(re^{-j\omega_0})^n u[n] \overset{z}{\longleftrightarrow} \frac{\frac{1}{2}}{1-re^{-j\omega_0}z^{-1}}, \qquad |z| > r.$$

From the linearity property, it follows that

$$\begin{aligned} X(z) &= \frac{\frac{1}{2}}{1-re^{j\omega_0}z^{-1}} + \frac{\frac{1}{2}}{1-re^{-j\omega_0}z^{-1}}, \qquad |z| > r \\ &= \frac{(1-r\cos\omega_0 z^{-1})}{1-2r\cos\omega_0 z^{-1} + r^2 z^{-2}}, \qquad |z| > r. \end{aligned} \tag{3.62}$$

Μετασχηματισμός z

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Example 3.17 Second-Order Pole

As another example of the use of the differentiation property, let us determine the z-transform of the sequence

$$x[n] = na^n u[n] = n(a^n u[n]).$$

From the z-transform pair of Example 3.1 and the differentiation property, it follows that

$$\begin{aligned} X(z) &= -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right), \qquad |z| > |a| \\ &= \frac{az^{-1}}{(1-az^{-1})^2}, \qquad |z| > |a|. \end{aligned}$$

Therefore,

$$na^n u[n] \overset{z}{\longleftrightarrow} \frac{az^{-1}}{(1-az^{-1})^2}, \qquad |z| > |a|.$$

Example 3.18 Time-Reversed Exponential Sequence

As an example of the use of the property of time reversal, consider the sequence

$$x[n] = a^{-n}u[-n],$$

which is a time-reversed version of $a^n u[n]$. From the time-reversal property, it follows that

$$X(z) = \frac{1}{1-az} = \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}}, \qquad |z| < |a^{-1}|.$$

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Example 3.19 Evaluating a Convolution Using the z-Transform

Let $x_1[n] = a^n u[n]$ and $x_2[n] = u[n]$. The corresponding z-transforms are

$$X_1(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|,$$

and

$$X_2(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}}, \qquad |z| > 1.$$

If $|a| < 1$, the z-transform of the convolution of $x_1[n]$ with $x_2[n]$ is then

$$Y(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})} = \frac{z^2}{(z-a)(z-1)}, \qquad |z| > 1. \qquad (3.64)$$

The poles and zeros of $Y(z)$ are plotted in Figure 3.14, and the region of convergence is seen to be the overlap region. The sequence $y[n]$ can be obtained by determining the inverse z-transform. Expanding $Y(z)$ in Eq. (3.64) in a partial fraction expansion, we get

$$Y(z) = \frac{1}{1-a}\left(\frac{1}{1 - z^{-1}} - \frac{a}{1 - az^{-1}}\right), \qquad |z| > 1.$$

Therefore,

$$y[n] = \frac{1}{1-a}(u[n] - a^{n+1}u[n]).$$

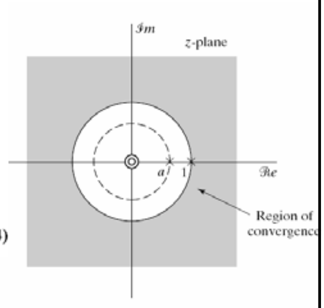

Figure 3.14 Pole–zero plot for the z-transform of the convolution of the sequences $u[n]$ and $a^n u[n]$.