ECE 631
System Theory

II. Linear Spaces

Functions

Given two sets X and Y, by the **function** $f:\mathrm{X}\mapsto\mathrm{Y}$ it is meant that for every $x\in\mathrm{X}$ there is assigned one and only one element $y\in\mathrm{Y}$, denoted by $f\left(x\right)$.

Range of f: $f\left(\mathrm{X}\right)=\left\{f\left(x\right)\middle|x\in\mathrm{X}\right\}$

Image of $V\subset\mathrm{X}$: $f\left(V\right)=\left\{f\left(x\right)\middle|x\in V\right\}$

"function" : "map", "operator", "transformation"

Examples ….

Properties of Functions

- $f : \mathrm{X} \mapsto \mathrm{Y}$ is **<u>one-to-one</u>** (1-1) (or **<u>injective</u>**)

  if $f\left(x_1\right) = f\left(x_2\right) \Rightarrow x_1 = x_2$

  [or $\left(x_1 \neq x_2 \Rightarrow f\left(x_1\right) \neq f\left(x_2\right)\right)$]

- $f : \mathrm{X} \mapsto \mathrm{Y}$ is **<u>onto</u>** (or **<u>surjective</u>**) if $f\left(\mathrm{X}\right) = \mathrm{Y}$

- $f : \mathrm{X} \mapsto \mathrm{Y}$ is one-to-one and onto (or **<u>bijective</u>**)

  if f is both surjective and injective

Examples ….

Linear Space (Linear Vector Space)

Linear Vector Space: (X, F)

For our purposes the scalar field F is always either $\mathbb{R}$ or $\mathbb{C}$

Definition: A set X is called a linear space over the field F if the following axioms are satisfied:

(A) For any $x, y \in \mathrm{X}$, the sum is defined and is in X; the sum is denoted by $x + y$.

(A1) $x + y = y + x$ (commutativity)

(A2) $(x + y) + z = x + (y + z)$ (associativity)

(A3) there exists an element $0 \in \mathrm{X}$ called the zero vector s.t. $x + 0 = x$ for all $x \in \mathrm{X}$

(A4) For every $x \in \mathrm{X}$ there is an element $(-x) \in \mathrm{X}$ such that $x + (-x) = 0$

Linear Space (Linear Vector Space)

Definition (continued):

(SM) For each scalar $\alpha \in F$ and each vector $x \in \mathrm{X}$ the operation of scalar multiplication is defined and denoted by $\alpha \cdot x \in \mathrm{X}$

(SM1) $(\alpha\beta) \cdot x = a \cdot (\beta \cdot x) \quad \alpha, \beta \in F,\ x \in \mathrm{X}$

(SM2) $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y \quad \alpha \in F,\ x, y \in \mathrm{X}$

(SM3) $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x \quad \alpha, \beta \in F,\ x \in \mathrm{X}$

(SM4) with $1 \in F$ being the multiplicative identity

$$1 \cdot x = x \quad \forall x \in \mathrm{X}$$

Linear Space (Examples)

<u>Example 1:</u> $\left(\mathbb{R}^n, \mathbb{R}\right)$ is a linear space.

<u>Example 2:</u> $\left(\mathbb{C}^n, \mathbb{C}\right)$ is a linear space.

<u>Example 3:</u> $\left(C\left[0, T\right], \mathbb{R}\right)$ is a linear space.

set of all continuous time functions $f\left(t\right)$
defined in the interval $0 \le t \le T$

Linear Space (Examples)

Example 4: $\left(P_n, \mathbb{R}\right)$ is a linear space.

set of all polynomials of degree less than or equal to n

Example 5:

$$f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix} \qquad P_n^{2\times 2} = \begin{bmatrix} P_n^{11} & P_n^{12} \\ P_n^{21} & P_n^{22} \end{bmatrix}$$

Linear Space (Examples)

Two examples of linear spaces are very important: (i) real spaces; (ii) function spaces.

Real Spaces: $\left(\mathbb{R}^n, \mathbb{R}\right)$ or more generally $\left(\mathbb{C}^n, \mathbb{C}\right)$

$$x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\top \in \mathbb{R}^n$$
$$y = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}^\top \in \mathbb{R}^n$$

(A) $x + y = \begin{bmatrix} x_1 + y_1 & x_2 + y_2 & \cdots & x_n + y_n \end{bmatrix}^\top \in \mathbb{R}^n$

(SM) $\alpha x = \begin{bmatrix} \alpha x_1 & \alpha x_2 & \cdots & \alpha x_n \end{bmatrix}^\top \in \mathbb{R}^n$

Linear Space (Examples)

Function Spaces:

$$f(s) = \begin{bmatrix} f_1(s) \\ \vdots \\ f_n(s) \end{bmatrix} \qquad g(s) = \begin{bmatrix} g_1(s) \\ \vdots \\ g_n(s) \end{bmatrix} \qquad s \in \mathcal{D}$$

(A) $$(f+g)(s) = \begin{bmatrix} f_1(s) + g_1(s) \\ \vdots \\ f_n(s) + g_n(s) \end{bmatrix} = f(s) + g(s)$$

(SM) $$(\alpha f)(s) = \begin{bmatrix} \alpha f_1(s) \\ \vdots \\ \alpha f_n(s) \end{bmatrix} = \alpha f(s)$$

Linear Space (Examples)

More generally: Let $(\mathrm{X},\mathbb{R})$ be a linear space. Let $\mathcal{D}$ be a set and $\mathcal{F}$ the class of functions that map $\mathcal{D}$ into X.

$$\mathcal{F}=\left\{ f \,\middle|\, f:\mathcal{D}\mapsto \mathrm{X}\right\}$$

on $\mathcal{F}$ define addition: $(f+g)(s)=f(s)+g(s) \qquad f,g\in\mathcal{F},\ s\in\mathcal{D}$

and scalar multiplication: $(\alpha f)(s)=\alpha f(s) \quad \alpha\in\mathbb{R},\ f\in\mathcal{F}, s\in\mathcal{D}$

Linear Subspaces

Let $(\mathrm{X}, \mathbb{R})$ be a linear space and Y a subset of X; i.e., $\mathrm{Y} \subset \mathrm{X}$

<u>**Definition:**</u> $(\mathrm{Y}, \mathbb{R})$ is a linear subspace of X if

- $x_1, x_2 \in \mathrm{Y} \Rightarrow x_1 + x_2 \in \mathrm{Y}$
- $\alpha \in \mathbb{R}, x \in \mathrm{Y} \Rightarrow \alpha x \in \mathrm{Y}$

<u>Note:</u> To check whether $(\mathrm{Y}, \mathbb{R})$ is a linear subspace of X

all we have to check is: $\alpha x_1 + x_2 \in \mathrm{Y} \quad \forall x_1, x_2 \in \mathrm{Y}, \forall \alpha \in \mathbb{R}$

(Verify!)

Linear Subspaces (Examples)

Example:

Consider $C\left[0,T\right]=\left\{ f \,\middle|\, f:\left[0,T\right]\mapsto \mathbb{R}\right\}$, f : continuous

linear space of continuous functions

Let $\mathcal{M}=\left\{ f \,\middle|\, f \in C\left[0,T\right], f\left(0\right)=0\right\}$

Is $\mathcal{M}$ is subspace of $C\left[0,T\right]$?

- $\mathcal{M}\subset C\left[0,T\right]$ (by definition)
- For any $f_1, f_2 \in \mathcal{M}, \alpha \in \mathbb{R}$

$$(\alpha f_1 + f_2)\left(0\right)=\alpha f_1\left(0\right)+ f_2(0)=\alpha\cdot 0+0=0$$

$\Rightarrow \alpha f_1 + f_2 \in \mathcal{M}$ $\Rightarrow \mathcal{M}$ is a subspace of $C\left[0,T\right]$

Linear Subspaces (Examples)

Example:

Let Y_1, Y_2 be subspace of X .

Then $Y_1 \cap Y_2$ is also a subspace of X.

Proof: Let $f_1, f_2 \in Y_1 \cap Y_2,\ \alpha \in \mathbb{R}$

Then $\alpha f_1 + f_2 \in Y_1$ (since $f_1, f_2 \in Y_1$)

Similarly $\alpha f_1 + f_2 \in Y_2$

$\Rightarrow \alpha f_1 + f_2 \in Y_1 \cap Y_2$

How to prove that $\mathcal{M}$ is a subspace? How to prove that it is not?

Linear Subspaces (Examples)

Example:

Is $\mathrm{X} = \left\{ x \in \mathbb{R}^3 \middle| x_2 = 1 \right\}$ a linear subspace of $\left(\mathbb{R}^3, \mathbb{R}\right)$?

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ be arbitrary vectors in X.

We want to check if $\alpha x + y \in \mathrm{X}, \ \forall \alpha \in \mathbb{R}$

Let

$$z := \alpha x + y = \begin{bmatrix} \alpha x_1 + y_1 \\ \alpha x_2 + y_2 \\ \alpha x_3 + y_3 \end{bmatrix} = \begin{bmatrix} \alpha x_1 + y_1 \\ \alpha + 1 \\ \alpha x_3 + y_3 \end{bmatrix} \Rightarrow z \notin \mathrm{X}, \ \forall \alpha \in \mathbb{R}$$

Example: Is $\mathrm{X}_1 = \left\{ x \in \mathbb{R}^3 \middle| x_2 = 0 \right\}$ a linear subspace of $\left(\mathbb{R}^3, \mathbb{R}\right)$

➔ Yes. (verify!)

Linear Independence

Definition: **(Linear dependence)**

Let $(\mathrm{X}, \mathbb{R})$ be a linear space. A finite number of vectors $\{x_i\} = \{x_1 \quad x_2 \quad \cdots \quad x_n\}$ are linearly dependent if there exist a set of n scalars α_i, at least one of which is not zero such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0_{\mathrm{X}}$$

the zero of the linear space

$$\sum_{i=1}^{n} \alpha_i x_i = 0_{\mathrm{X}}$$

Definition: **(Linear Independence)**

Let $(\mathrm{X}, \mathbb{R})$ be a linear space. The set of vectors $\{x_i\}$ is linearly independent if $\sum_{i=1}^{n} \alpha_i x_i = 0 \quad \Rightarrow \quad \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$

Linear Independence

Fact:

If $\{x_i\}$ is linearly dependent, then at least one of the vectors can be written as a linear combination of the others.

Proof:

Assume with loss of generality that $\alpha_1 \neq 0$

$$\Rightarrow x_1 = -\frac{1}{\alpha_1}\left[a_2 x_2 + \cdots + a_n x_n\right].$$

Linear Independence (Examples)

Example:

Is the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix} \right\}$ linearly independent?

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix} = 0 \Rightarrow \begin{matrix} \alpha_1 - \alpha_3 = 0 \\ \alpha_2 - \alpha_3 = 0 \\ 2\alpha_1 - 2\alpha_3 = 0 \end{matrix}$$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3$. Let $\alpha_1 = \alpha_2 = \alpha_3 = 1$

$\Rightarrow$ the set in linearly dependent

Linear Independence (Examples)

Example:

Is the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ linearly independent ?

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 \Rightarrow \begin{matrix} \alpha_1 = 0 \\ \alpha_2 = 0 \\ 2\alpha_1 = 0 \end{matrix} \qquad \Rightarrow \alpha_1 = \alpha_2 = 0.$$

$\Rightarrow$ the set in linearly independent.

Linear Independence (Examples)

Example:

Do there exist values $x_1, x_2, x_3 \in \mathbb{R}$ such that the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\}$$ is linearly independent?

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} \alpha_1 + \alpha_4 x_1 = 0 \\ \alpha_2 + \alpha_4 x_2 = 0 \\ \alpha_3 + \alpha_4 x_3 = 0 \end{matrix} \Rightarrow \begin{matrix} \alpha_1 = -\alpha_4 x_1 \\ \alpha_2 = -\alpha_4 x_2 \\ \alpha_3 = -\alpha_4 x_3 \end{matrix}$$

$\Rightarrow$ always linearly dependent.

OR: For any x_1, x_2, x_3

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow$ the fourth vector is a linear combination of the other three

Linear Independence (Examples)

<u>Example:</u>

Let $\mathrm{X} = \left\{ \cos\left(\kappa \pi t\right) \middle| \kappa = 0,1,2,\ldots,n \right\} \cup \left\{ \sin\left(\kappa \pi t\right) \middle| \kappa = 1,2,\ldots,m \right\}$

Consider $\mathrm{Y} = \left\{ f \,\middle|\, f = \sum_{i=1}^{N} \alpha_i x_i, \alpha_i \in \mathbb{R}, x_i \in \mathrm{X} \right\}$ that is, Y is composed of all linear combination of elements of X.

→ Y is a linear space (verify!)

$N = n + m + 1$

→ **Is X a linearly independent set of vectors?**

Suppose that $\sum_{i=1}^{N} \alpha_i x_i = 0$; we must show that this implies

$\alpha_1 = \alpha_2 = \ldots = \alpha_N = 0$ for X to be linearly independent.

$$\Rightarrow \alpha_1 \cos\left(0\pi t\right) + \alpha_2 \cos\left(\pi t\right) + \ldots + \alpha_n \cos\left(n\pi t\right) + \ldots + \alpha_N \sin\left(m\pi t\right) = 0$$

Linear Independence (Examples)

Example (continued)

$$\alpha_1 \cos(0\pi t) + \alpha_2 \cos(\pi t) + \ldots + \alpha_n \cos(n\pi t) + \ldots + \alpha_N \sin(m\pi t) = 0$$

- Integrate both sides from -1 to 1 gives $\alpha_1 = 0$
- Multiply both sides by $\cos(\pi t)$ and integrate from -1 to 1 gives $\alpha_2 = 0$
- Multiply both sides by $\cos(2\pi t)$ and integrate from -1 to 1 gives $\alpha_3 = 0$

Continue this to get $\alpha_1 = \alpha_2 = \ldots = \alpha_N = 0$

→ X is a linearly independent set of vectors

Span, Basis, Dimension

Definition: Let $\mathrm{Y} = \{y_i : i = 1, \ldots, n\}$ be a subset of a linear space $(\mathrm{X}, \mathbb{R})$. The collection of linear combinations of vectors in Y is called **span** of Y, denoted as: $sp(\mathrm{Y})$

$$sp(\mathrm{Y}) = \left\{ x \in \mathrm{X} \,\middle|\, x = \sum_{i=1}^{n} \alpha_i y_i, \alpha_i \in \mathbb{R}, y_i \in \mathrm{Y} \right\}$$

Span, Basis, Dimension

Definition (Basis): Let $\mathrm{Y} = \{y_i : i = 1, \ldots, n\}$ be a set of vectors in $(\mathrm{X}, \mathbb{R})$. The set Y is called a **basis** for X if

(i) the vectors $\{y_i\}$ are linearly independent.

(ii) $sp(\mathrm{Y}) = \mathrm{X}$

$(\mathrm{X}, \mathbb{R})$

- $\{y_i\}$ are called basis vectors of X
- Are $\{y_i\}$ unique basis vectors?

Span, Basis, Dimension

Note: If $\{y_i\}$ are the basis vectors of X then for any $x \in \mathrm{X}$ there exists scalars $\alpha_1, \ldots, \alpha_n$ such that $x = \sum_{i=1}^{n} \alpha_i y_i$

Fact: This parametrization is unique.

Proof: Suppose $\exists \beta_1, \beta_2, \ldots, \beta_n$ such that $x = \sum_{i=1}^{n} \beta_i y_i$; $x = \sum_{i=1}^{n} \alpha_i y_i$

$x - x = 0 = \sum_{i=1}^{n} (\beta_i - \alpha_i) y_i$. Since $\{y_i\}$ are linearly independent:

$\beta_1 - \alpha_1 = 0;\ \beta_2 - \alpha_2 = 0;\ \ldots\ \beta_n - \alpha_n = 0; \Rightarrow \alpha_1 = \beta_1 \ldots \Rightarrow$ uniqueness.

Definition (Dimension): If a basis Y for $(\mathrm{X}, \mathbb{R})$ has n elements then X is an n-dimensional linear vector space.

Span, Basis, Dimension

<u>Example</u>: $(\mathbb{R}^n, \mathbb{R})$ has a basis $\{e_1, e_2, \ldots, e_n\}$ where

$e_i = [0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0]^\top$. $\mathbb{R}^n$ is n-dimensional space.

i-th position

<u>Example</u>: Let $\mathrm{Y} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Then:

- $sp(\mathrm{Y}) = \mathbb{R}^3$ (verify!)
- Y is a basis for $\mathbb{R}^3$.

Infinite-Dimensional Linear Vector Spaces

- These results can be extended to infinite-dimensional linear vector spaces
- For example: $C\left[0,T\right]$ is an infinite-dimensional linear vector space.
- $M=\left\{\cos\left(\kappa\pi t\right),\sin\left(\kappa\pi t\right);\kappa=0,1,2,\ldots\right\}$ $\quad \mathrm{X}=sp\left(M\right)$
  M is a basis for the infinite-dimensional space X.

Dimension of Linear Vector Spaces (Examples)

1) $\left(\mathbb{R}^{3\times 2}, \mathbb{R}\right)$ $\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}$ 3x2 matrices of reals.

2) Polynomials of degree 4 $\left(P_4, \mathbb{R}\right)$
e.g. $x^4 + 3x^3 + x^2 + 4x + 1.$

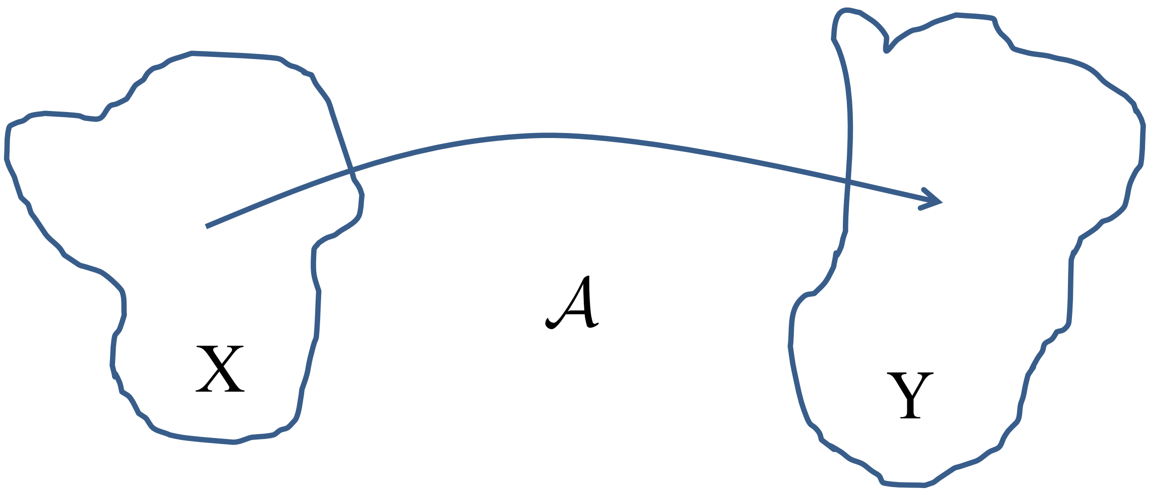
3) $\left(\mathbb{C}, \mathbb{R}\right)$

4) $\left(\mathbb{C}, \mathbb{C}\right)$

Linear Transformations

<u>Definition</u>: Let $\mathrm{A}: \mathrm{X} \mapsto \mathrm{Y}$ where X, Y are linear vector spaces over the same field $\mathcal{F}$. $\mathcal{A}$ Is a <u>linear transformation</u> or <u>operator</u> if

$$\mathcal{A}(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 \mathcal{A}(x_1) + \alpha_2 \mathcal{A}(x_2) \ \forall \alpha_1, \alpha_2 \in \mathcal{F}, \ x_1, x_2 \in \mathrm{X}$$



- $\mathcal{A}(x_1 + x_2) = \mathcal{A}(x_1) + \mathcal{A}(x_2)$ Additivity.
- $\mathcal{A}(\alpha x) = \alpha \mathcal{A}(x)$ Homogeneity
- Together: $\mathcal{A}(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 \mathcal{A}(x_1) + \alpha_2 \mathcal{A}(x_2)$

  <u>SUPERPOSITION</u>.

Linear Transformations

Example: $\mathrm{A}:\mathbb{R}^3 \mapsto \mathbb{R}^3$ $\qquad y = \mathrm{A}x$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 1 \\ -1 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

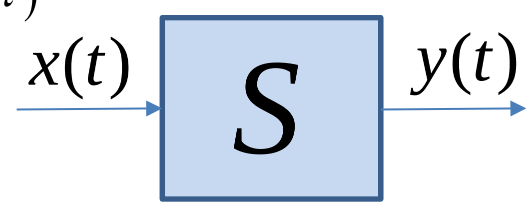
By the standard rules of matrix addition and scalar multiplication, we have that the matrix is a linear transformation or linear operator.

Example: (convolution) Let $\mathrm{X} = PC[0,\infty]$, $Y = PC[0,\infty]$ and $\mathcal{A}$ be defined as

linear space of piecewise continuous functions

$$y(t) = (\mathcal{A}x)(t) = \int_0^t \underbrace{\exp(-(t-\tau))}_{\text{more general: } h(t-\tau)} x(\tau)\,d\tau$$

➔ Convolution is a linear operator:



Null & Range Spaces of Linear Operators

Definition: Let $\mathcal{A}$ be a linear operator, $\mathcal{A} : \mathrm{X} \mapsto \mathrm{Y}$.

The set $\mathcal{N}(\mathcal{A}) = \left\{ x \in \mathrm{X} \,\middle|\, \mathcal{A}(x) = 0_{\mathrm{Y}} \right\}$ is called the null space of $\mathcal{A}$.

The set $\mathcal{R}(\mathcal{A}) = \left\{ y \in \mathrm{Y} \,\middle|\, y = \mathcal{A}(x), x \in \mathrm{X} \right\}$ is called the range space of $\mathcal{A}$.

Fact: $\mathcal{N}(\mathcal{A})$ is a linear subspace of X.
$\mathcal{R}(\mathcal{A})$ is a linear subspace of Y.
(Verify!)