ECE 631
System Theory

III. Matrix Representations

Matrix Representations

Some observations:

- A matrix can be considered as a linear operator.
- Not every operator can be represented by a matrix. e.g. the convolution operator on $PC[0,\infty]$ into $PC[0,\infty]$ cannot be considered in matrix form.
- If the linear operator $\mathcal{A}: \mathrm{X} \mapsto \mathrm{Y}$, where the dimensions of X and Y are finite, then $\mathcal{A}$ can always be represented as a matrix with respect to given bases of X and Y.

Matrix Representations

Consider $\mathcal{A}: \mathrm{X} \mapsto \mathrm{Y}$, where X, Y are finite dimensional linear vector spaces over the field $\mathbb{R}$. Let $\{x_1, \ldots, x_n\}$ be a basis for X .
Then for any $\chi \in \mathrm{X}$ there is a unique representation

$$\chi = \sum_{i=1}^{n} \alpha_i x_i \qquad [\chi]_x = (a_1, a_2, \ldots, \alpha_n)$$

$\{a_1, a_2, \ldots, \alpha_n\}$ are called the <u>components</u> of χ with respect to the basis $\{x_1, x_2, \ldots, x_n\}$.

Matrix Representations

i-th position

Special case: If $x_i = e_i = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^\top$ then:

natural Cartesian coordinates

$$\alpha_i = \chi_i,\ i = 1, 2, \ldots, n \qquad [\chi]_e = (\chi_1, \chi_2, \ldots, \chi_n)$$

Consider $\mathcal{A}(\chi)$; by linearity of $\mathcal{A}$:

$$\begin{aligned} \mathcal{A}(\chi) &= \mathcal{A}(\alpha_1 x_1 + \ldots + \alpha_n x_n) \\ &= \alpha_1 \mathcal{A}(x_1) + \ldots + \alpha_n \mathcal{A}(x_n) \\ &= \sum_{i=1}^{n} \alpha_i \mathcal{A}(x_i) \end{aligned}$$

$$\psi = \mathcal{A}(\chi) = \sum_{i=1}^{n} \alpha_i \mathcal{A}(x_i) \ , \quad \psi \in \mathrm{Y}$$

Matrix Representations

Let $\{y_1, y_2, \ldots, y_n\}$ be a basis for Y.

$$\boxed{\psi = \sum_{j=1}^{m} \beta_j y_j} \quad \text{and} \quad \boxed{\mathcal{A}(x_i) = \sum_{j=1}^{m} a_{ji} y_j}$$

$$\Rightarrow \psi = \sum_{j=1}^{m} \beta_j y_j = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} a_{ji} y_j = \sum_{j=1}^{m} \left[\sum_{i=1}^{n} a_{ji} \alpha_i \right] y_j$$

$$\Rightarrow \beta_j = \sum_{i=1}^{n} a_{ji} \alpha_i \quad , \quad j = 1, \ldots, m.$$

Let $[\mathcal{A}]_{y,x} = \underbrace{\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{bmatrix}}_{\text{n columns}} \left.\vphantom{\begin{bmatrix} \alpha \\ \vdots \\ \alpha \end{bmatrix}}\right\}$ m rows

$$\Rightarrow [\mathcal{A}(\chi)]_y = [\mathcal{A}]_{y,x} [\chi]_x$$

$$\boxed{[\psi]_y = A[\chi]_x}$$

where $A = [\mathcal{A}]_{y,x}$

Matrix Representations

To summarise:

Theorem: Let $\left(\mathrm{X}, \mathcal{F} \right)$ have $\left\{ x_1, x_2, \ldots, x_n \right\}$ as a basis; let $\left(\mathrm{Y}, \mathcal{F} \right)$ have $\left\{ y_1, y_2, \ldots, y_m \right\}$ as a basis; let $\mathcal{A} : \mathrm{X} \mapsto \mathrm{Y}$ be a linear operator. Then with respect to these bases, the operator $\mathcal{A}$ is represented by the $m \times n$ matrix

$$A = \left(\alpha_{ji} \right) = \left[\mathcal{A} \right]_{y,x}$$

where the elements of the i-th column of A are the components of $\mathcal{A}\left(x_i \right)$ with respect to the basis $\left\{ y_1, y_2, \ldots, y_m \right\}$.

Note:

- A matrix is just a representation of an operator.
- If we don't know the bases used, we don't really know the operator.

Matrix Representations

<u>Example</u>: $\mathcal{A} : \mathbb{R}^2 \mapsto \mathbb{R}^3$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longrightarrow \boxed{\mathcal{A}} \longrightarrow \begin{bmatrix} x_1 \\ -2x_1 \\ x_2 \end{bmatrix} \qquad \mathcal{A}(x) = \begin{bmatrix} x_1 \\ -2x_1 \\ x_2 \end{bmatrix}$$

(A) Basis for $\mathbb{R}^2$: $(u_1, u_2) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$

Basis for $\mathbb{R}^3$: $(v_1, v_2, v_3) = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$

Represent $\mathcal{A}$ in matrix with respect to the basis u, v.

$$\mathcal{A}(u_1) = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix}^\top = \alpha_1 v_1 + a_2 v_2 + \alpha_3 v_3 = 3v_1 - 2v_2 + 0v_3$$

$$\mathcal{A}(u_2) = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}^\top = \alpha_1 v_1 + a_2 v_2 + \alpha_3 v_3 = 3v_1 - 3v_2 + 1v_3$$

$$\boxed{[\mathcal{A}]_{v,u} = \begin{bmatrix} 3 & 3 \\ -2 & -3 \\ 0 & 1 \end{bmatrix}}$$

Matrix Representations

<u>Example (cont.)</u>: To test, consider $x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \alpha_1 u_1 + \alpha_2 u_2 = -2u_1 + 1u_2$

$$y = \mathcal{A}(x) = \begin{bmatrix} 3 & 3 \\ -2 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \quad \Rightarrow -3v_1 + 1v_2 + 1v_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

(B) New Basis:

Basis for $\mathbb{R}^2$: $(\tilde{u}_1, \tilde{u}_2) = \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right)$ Basis for $\mathbb{R}^3$: $(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3) = \left(\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$

What is $[\mathcal{A}]_{\tilde{v},\tilde{u}}$?

$$\mathcal{A}(\tilde{u}_1) = \begin{bmatrix} -1 & 2 & 0 \end{bmatrix}^\top = \alpha_1 \tilde{v}_1 + \alpha_2 \tilde{v}_2 + \alpha_3 \tilde{v}_3 = 1\tilde{v}_1 + 2\tilde{v}_2 + 0\tilde{v}_3$$

$$\mathcal{A}(\tilde{u}_2) = \begin{bmatrix} 0 & 0 & 3 \end{bmatrix}^\top = \alpha_1 \tilde{v}_1 + \alpha_2 \tilde{v}_2 + \alpha_3 \tilde{v}_3 = 0\tilde{v}_1 - 3\tilde{v}_2 + 3\tilde{v}_3$$

$$[\mathcal{A}]_{\tilde{v},\tilde{u}} = \begin{bmatrix} 1 & 0 \\ 2 & -3 \\ 0 & 3 \end{bmatrix}$$

Matrix Representations

Example (cont.):

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \longrightarrow \boxed{\mathcal{A}} \longrightarrow \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \quad \mathcal{A}(\chi)$$

$$[\chi]_{\tilde{u}}\ ?$$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} -1 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 2 & -3 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$[\mathcal{A}(\chi)]_{\tilde{v}}\ ? \quad \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = 1\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{[\mathcal{A}(\chi)]_{\tilde{v}} = [\mathcal{A}]_{\tilde{v},\tilde{u}}[\chi]_{\tilde{u}}}$$

Matrix Representations

<u>Example</u>: $\mathcal{A} : C^1[0,\ 1] \mapsto C[0,\ \ 1]$

Space of continuously differentiable on [0 1]

$$\mathcal{A}(f) = \frac{d}{dt} f$$ (differential operator)

$\mathcal{A}$ is a linear operator (check!)

(A) Consider $M = sp(1, t, t^2, t^3)$; then $\mathcal{A}(M) \subset M$

Restrict $\mathcal{A}$: $\mathcal{A} : M \mapsto M$ $\dim(M) = 4 < \infty$

Let $\{1, t, t^2, t^3\}$ be the basis in both the domain and range.

What is $[\mathcal{A}]_{\mu,\mu}$?

$\mathcal{A}(\mu_1) = \mathcal{A}(1) = 0$ Components : $(0,0,0,0)$

$\mathcal{A}(\mu_2) = \mathcal{A}(t) = 1$ Components : $(1,0,0,0)$

$\mathcal{A}(\mu_3) = \mathcal{A}(t^2) = 2t$ Components : $(0,2,0,0)$

$\mathcal{A}(\mu_4) = \mathcal{A}(t^4) = 3t^2$ Components : $(0,0,3,0)$

Matrix Representations

Example (continued):

$$[\mathcal{A}]_{\mu,\mu} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Check! $\mathcal{A}\left(3+7t+t^2-5t^3\right) = 7+2t-15t^2$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ -15 \\ 0 \end{bmatrix} = 7\cdot 1 + 2\cdot t + (-15)t^2 + 0\cdot t^3$$

(B) Consider $N = sp\left(1, t, t^2\right)$ $\mathcal{A} : M \mapsto N$

Domain basis: $\mu = \left\{1, t, t^2, t^3\right\}$

Range basis: $\lambda = \left\{1, t, t^2\right\}$

$$[\mathcal{A}]_{\lambda,\mu} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Matrix Representation and Change of Basis

$\mathcal{A}: \mathrm{X} \mapsto \mathrm{Y}$ linear operator.

$x = \{x_1, \ldots, x_n\}$ Basis for X Matrix $[\mathcal{A}]_{y,x}$: representation for $\mathcal{A}$

$y = \{y_1, \ldots, y_n\}$ Basis for Y

$\tilde{x} = \{\tilde{x}_1, \ldots, \tilde{x}_n\}$ "new" basis of X

$\tilde{y} = \{\tilde{y}_1, \ldots, \tilde{y}_n\}$ "new" basis of Y

$[\mathcal{A}]_{\tilde{y},\tilde{x}}$ is another representation for $\mathcal{A}$

➔ How are $[\mathcal{A}]_{y,x}$ and $[\mathcal{A}]_{\tilde{y},\tilde{x}}$ related?

$$\tilde{x}_i = \sum_{l=1}^{n} p_{li} x_l \ , \quad i = 1, \ldots, n \qquad P = (p_{li}) \qquad n \times n \text{ matrix}$$

i-th column of P consists of the components of $\tilde{x}_i$ with respect to $x = \{x_1, \ldots, x_n\}$.

Matrix Representation and Change of Basis

For any $\chi \in \mathrm{X} : \chi = \sum_{i=1}^{n} \alpha_i x_i$ $\qquad$ $\chi = \sum_{i=1}^{n} \tilde{\alpha}_i \tilde{x}_i = \sum_{i=1}^{n} \tilde{\alpha}_i \sum_{l=1}^{n} p_{li} x_l$

$$= \sum_{i=1}^{n} \sum_{l=1}^{n} \tilde{\alpha}_i p_{li} x_l = \sum_{l=1}^{n} \alpha_l x_l$$

$$\Rightarrow \alpha_l = \sum_{i=1}^{n} p_{li} \tilde{\alpha}_i \ \ ; \ l = 1, \ldots, n.$$

$$\boxed{[\chi]_x = P[\chi]_{\tilde{x}} \Rightarrow [\chi]_{\tilde{x}} = P^{-1}[\chi]_x}$$

Similarly: $\tilde{y}_i = \sum_{\kappa=1}^{n} q_{\kappa i} y_\kappa \ ; \ i = 1, \ldots, m.$

$Q = (q_{ki})$ $\quad$ $m \times m$ matrix

i-th column of Q consists of the components of $\tilde{y}_i$ with respect to $y = \{y_1, \ldots, y_n\}$.

Matrix Representation and Change of Basis

For any $\psi \in \mathrm{Y} : \psi = \sum_{\kappa=1}^{m} \beta_\kappa y_\kappa$

$$\psi = \sum_{i=1}^{m} \tilde{\beta}_i \tilde{y}_i = \sum_{i=1}^{m} \tilde{\beta}_i \sum_{\kappa=1}^{m} q_{\kappa i} y_\kappa = \sum_{\kappa=1}^{m} \sum_{i=1}^{m} \tilde{\beta} q_{\kappa i} y_\kappa = \sum_{\kappa=1}^{m} \beta_\kappa y_\kappa$$

$$\Rightarrow \beta_k = \sum_{i=1}^{m} q_{\kappa i} \tilde{\beta}_i \quad , \quad \kappa = 1, \ldots, m.$$

$$\boxed{[\psi]_y = Q[\psi]_{\tilde{y}} \;\; ; \;\; [\psi]_{\tilde{y}} = Q^{-1}[\psi]_y}$$

Let $A = [\mathcal{A}]_{y,x} \; ; \; \tilde{A} = [\mathcal{A}]_{\tilde{y},\tilde{x}}$

$$[\psi]_y = [\mathcal{A}(\chi)]_y = A[\chi]_x = AP[\chi]_{\tilde{x}}$$

$$\Rightarrow Q[\mathcal{A}(\chi)]_{\tilde{y}} = Q\tilde{A}[\chi]_{\tilde{x}} = AP[\chi]_{\tilde{x}}$$

$$\tilde{A}[\chi]_{\tilde{x}} = Q^{-1} AP[\chi]_{\tilde{x}}$$

$$\Rightarrow \quad \boxed{\tilde{A} = Q^{-1} AP} \quad \boxed{A = Q\tilde{A}P^{-1}}$$

Matrix Representation and Change of Basis

From previous example: $A = [\mathcal{A}]_{v,u} = \begin{bmatrix} 3 & 3 \\ -2 & -3 \\ 0 & 1 \end{bmatrix}$ $\quad \tilde{A} = [\mathcal{A}]_{\tilde{v},\tilde{u}} = \begin{bmatrix} 1 & 0 \\ 2 & -3 \\ 0 & 3 \end{bmatrix}$

$$\left.\begin{aligned} \tilde{u}_1 &= \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \tilde{u}_2 &= \begin{bmatrix} 0 \\ 3 \end{bmatrix} = -3 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}\right\} \Rightarrow P = \begin{bmatrix} -1 & -3 \\ 0 & 3 \end{bmatrix}$$

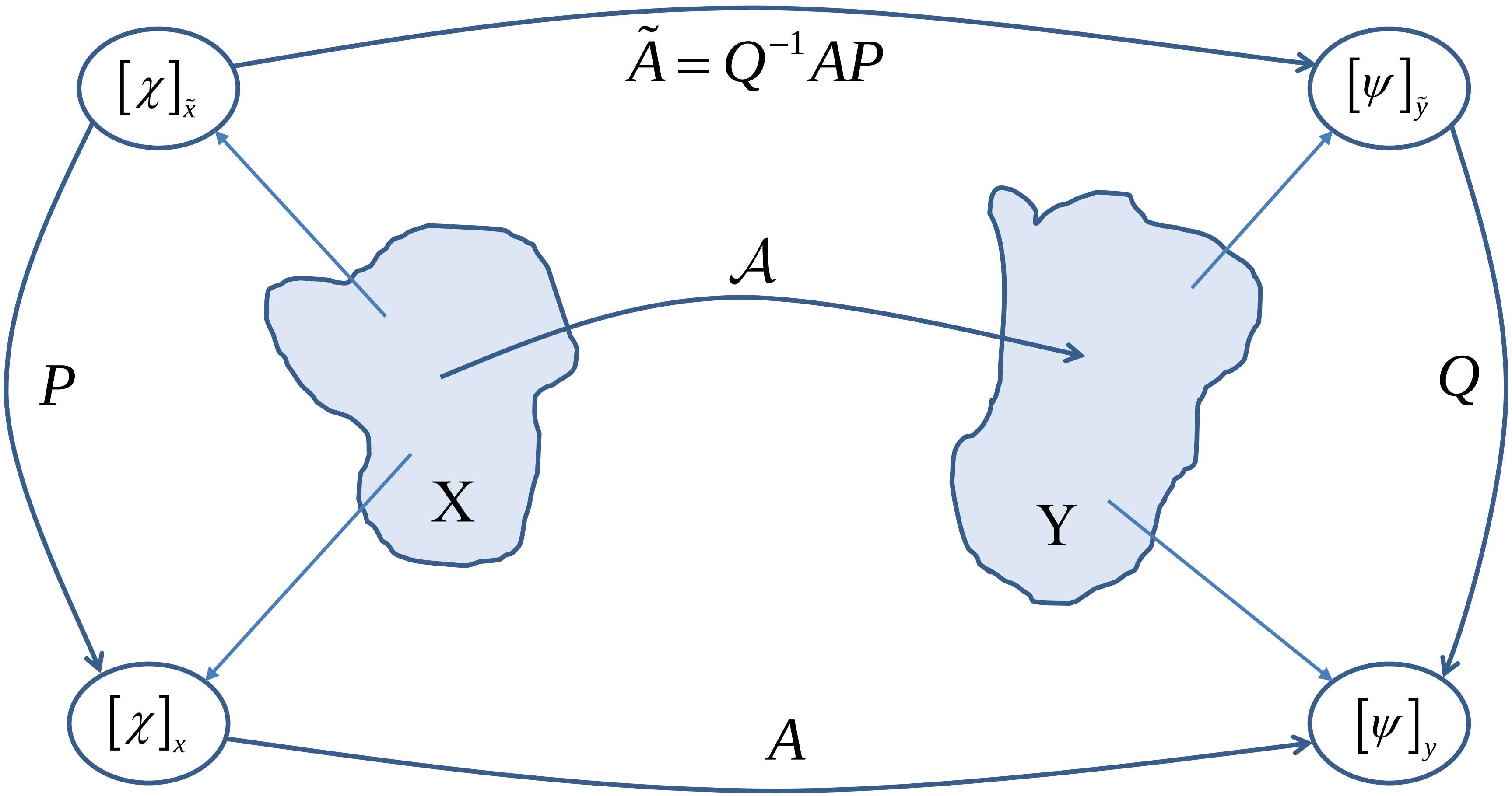
$$\tilde{A} = Q^{-1} A P \qquad A = Q \tilde{A} P^{-1}$$

(Check !)

$$\left.\begin{aligned} \tilde{v}_1 &= \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ \tilde{v}_2 &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ \tilde{v}_3 &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{aligned}\right\} \Rightarrow Q = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix Representation and Change of Basis

Special case: $\mathcal{A} : \mathrm{X} \mapsto \mathrm{X}$ $\boxed{\tilde{A} = P^{-1}AP}$



Relation between Range and Null Space

$\mathcal{A}:(\mathrm{X},\mathcal{F})\mapsto(\mathrm{Y},\mathcal{F})$ linear operator

$\dim(\mathrm{X})=n$ $\quad\mathcal{R}(\mathcal{A})\subset\mathrm{Y}$ Range Space

$\dim(\mathrm{Y})=m$ $\quad\mathcal{N}(\mathcal{A})\subset\mathrm{X}$ Null Space

$\boxed{\text{Fact: } \dim(\mathcal{R}(\mathcal{A}))+\dim(\mathcal{N}(\mathcal{A}))=n}$

dim(domain of $\mathcal{A}$)

Relation between Range and Null Space (Examples)

Example: $\mathcal{A}:\mathbb{R}^2 \mapsto \mathbb{R}^3$

$$\mathcal{A}(x)=\begin{bmatrix} x_1 \\ -2x_1 \\ x_2 \end{bmatrix}$$

Same as previous example

What is $\mathcal{N}(\mathcal{A})$?

$$\mathcal{A}(x)=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = x_2 \Rightarrow x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\dim\left(\mathcal{N}(\mathcal{A})\right)=0.$$

What is $\mathcal{R}(\mathcal{A})$? All 3-dimensional vectors of the form $\begin{bmatrix} \alpha_1 \\ -2\alpha_1 \\ \alpha_2 \end{bmatrix}$ where $\alpha_1, \alpha_2 \in \mathbb{R}$.

$$\begin{bmatrix} \alpha_1 \\ -2\alpha_1 \\ \alpha_2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis for $\mathcal{R}(\mathcal{A})$

$$\Rightarrow \dim\left(\mathcal{R}(\mathcal{A})\right)=0.$$

Relation between Range and Null Space (Examples)

Example: $\mathcal{A}:\mathbb{R}^4 \mapsto \mathbb{R}^4$

$$x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad \mathcal{A}(x)=\begin{bmatrix} 0 \\ 0 \\ x_1 \\ x_2+x_4 \end{bmatrix}$$

What is $\mathcal{N}(\mathcal{A})$?

$$\mathcal{A}(x)=\begin{bmatrix} 0 \\ 0 \\ x_1 \\ x_2+x_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1=0 \\ x_2+x_4=0 \end{matrix}$$

$$\mathcal{N}(\mathcal{A})=\left\{ x\in\mathbb{R}^4 \,\middle|\, x=\begin{bmatrix} 0 \\ \alpha \\ \beta \\ -\alpha \end{bmatrix}, \alpha,\beta\in\mathbb{R} \right\}$$

Relation between Range and Null Space(Examples)

Example: $\dim\left(\mathcal{N}(\mathcal{A})\right)$?

$$\begin{bmatrix} 0 \\ \alpha \\ \beta \\ -\alpha \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \dim\left(\mathcal{N}(\mathcal{A})\right) = 2.$$

$\dim\left(\mathcal{R}(\mathcal{A})\right)$? $\quad \mathcal{R}(\mathcal{A}) = \left\{ x \in \mathbb{R}^4 \,\middle|\, x = \begin{bmatrix} 0 \\ 0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}, \alpha_1, \alpha_2 \in \mathbb{R} \right\}$

$$\begin{bmatrix} 0 \\ 0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \dim\left(\mathcal{R}(\mathcal{A})\right) = 2.$$

Rank

Definition: The **rank** of the operator $\mathcal{A}$ is $\dim\left(\mathcal{R}(\mathcal{A})\right)$.

Fact: $rank(\mathcal{A})$ = maximum number of linearly independent columns of A (where A is any matrix representation of $\mathcal{A}$).
= maximum number of linearly independent rows of A.

Sometimes denoted as $\rho(\mathrm{A})$

Example:

$$\mathrm{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 2 \\ 0 & 3 & 3 & 6 \end{bmatrix} \qquad \rho(\mathrm{A}) = 2$$

Rank (Examples)

Example: (hard for a computer to find the rank).

$$\mathrm{A} = \begin{bmatrix} 1 & -1 \\ 0 & \varepsilon \end{bmatrix} \quad \rho(\mathrm{A}) = \begin{cases} 1 & \text{if } \varepsilon \neq 0 \\ 2 & \text{if } \varepsilon = 0 \end{cases}$$

Fact: $\mathrm{A} \in \mathbb{R}^{m \times n}$: $\qquad \rho(\mathrm{A}) \leq \min\{m, n\}$