ECE 631
System Theory

IV. Linear Algebra and Norms

Spectral Theory

<u>Definition</u>: Let $\mathcal{A}:\mathbb{R}^n \mapsto \mathbb{R}^n$ be a linear operator. Any scalar $\lambda \in \mathcal{F}$ such that $\mathcal{A}(x) = \lambda x$, where $x \neq 0$, is called an **<u>eigenvalue</u>** of $\mathcal{A}$ and x an **<u>eigenvector</u>** of $\mathcal{A}$.

<u>Note</u>: Along the direction of an eigenvector x, the operator $\mathcal{A}$ simply multiplies the vector by a scalar λ.

<u>Note</u>: $\mathrm{A}x = \lambda x \Rightarrow (\mathrm{A} - \lambda I)x = 0$
Find the values of λ such that the null-space of $(\mathrm{A} - \lambda I)$ is not simply $x = 0$, i.e. the trivial solution.

<u>Note</u>: To find λ we solve $\boxed{\det(\mathrm{A} - \lambda I) = 0}$

Characteristic Equation

Spectral Theory (Examples)

<u>Example</u>: $\mathrm{A}=\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ $\det\left(\mathrm{A}-\lambda I\right)=\det\begin{bmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix}$

$$=(2-\lambda)(3-\lambda)-2$$

$$=\lambda^2-5\lambda+4$$

$$=(\lambda-1)(\lambda-4)$$

Eigenvalues: $\lambda_1=1 \quad \lambda_2=4.$

Eigenvectors: $(\mathrm{A}-\lambda_1 I)x=0 \quad \Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$x_1+x_2=0 \Leftrightarrow x_1=-x_2.$$

(i) $x=\begin{bmatrix} 1 \\ -1 \end{bmatrix};$ General solution: $\begin{bmatrix} \alpha \\ -\alpha \end{bmatrix}$ $\alpha\in\mathbb{R},\alpha\neq 0.$

Spectral Theory (Examples)

Example (continued):

$$(\mathrm{A}-\lambda_2 I)y=0 \quad \Rightarrow \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2y_1+y_2=0 \quad \Rightarrow y_2=2y_1.$$

(ii) $\quad y=\begin{bmatrix} 1 \\ 2 \end{bmatrix};$ $\qquad$ General solution: $\begin{bmatrix} \alpha \\ 2\alpha \end{bmatrix}$ $\qquad \alpha\in\mathbb{R},\alpha\neq 0.$

Spectral Theory (Examples)

<u>Example</u>: $A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$

$$\det\left(A - \lambda I\right) = \det\begin{bmatrix} 3-\lambda & 2 \\ -1 & 1-\lambda \end{bmatrix}$$

$$= (3-\lambda)(1-\lambda) + 2$$

$$= \lambda^2 - 4\lambda + 5$$

Eigenvalues: $\lambda_1 = 2 + j \quad \lambda_2 = 2 - j.$

Eigenvectors are also complex :

$$(A - \lambda I)x = 0 \qquad x = \left\{ \begin{bmatrix} 2 \\ -1 + j \end{bmatrix}, \begin{bmatrix} 2 \\ -1 - j \end{bmatrix} \right\}$$

Spectral Theory

$\mathrm{A} \in \mathbb{R}^{n\times n}$: $\det\left(\mathrm{A} - \lambda I\right)$ is a polynomial of degree n.

$$\Delta(\lambda) = \det\left(\mathrm{A} - \lambda I\right) = (-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0$$
$$= (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$$

$\Delta(\lambda)$ is called the **characteristic polynomial of A**.

- $\lambda_1, \lambda_2, \ldots, \lambda_n$ can be real, complex, or a combination of real and complex.
- $\lambda_1, \lambda_2, \ldots, \lambda_n$ can be all distinct or we can have $\lambda_i = \lambda_j$ for $i \neq j$.
- $\det(\mathrm{A}) = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n = c_0$
- Suppose λ_i is complex: $\lambda_i = \alpha_i + j\beta_i$. Then $\overline{\lambda_i} = \alpha_i - j\beta_i$ is also an eigenvalue.

Spectral Theory

<u>Trace</u>: Let $\mathrm{A} \in \mathbb{R}^{n \times n}$ $\quad \mathrm{Trace}(\mathrm{A}) = \sum_{i=1}^{n} \alpha_{ii}$

Sum of diagonal elements.

<u>Example</u>: $\mathrm{A} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ $\quad \mathrm{Trace}(\mathrm{A}) = 3 + 1 = 4.$

<u>Lemma</u>: $\mathrm{Trace}(\mathrm{A}) = \lambda_1 + \lambda_2 + \ldots + \lambda_n = \sum_{i=1}^{n} \lambda_i$

<u>Lemma</u>: Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct eigenvalues of A. Then any $x_1, x_2, \ldots, x_n$ of corresponding eigenvectors is linearly independent.

$$\mathrm{M} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

<u>Modal matrix</u>.

$$\mathrm{M}^{-1}\mathrm{AM} = \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Singular Value Decomposition (SVD)

Theorem: Any $m \times n$ matrix can be factored into:

$$A = U \Sigma V^{\top}$$

where

- U is an $m \times m$ orthogonal matrix, i.e., $U^{\top} U = I$
- V is an $n \times n$ orthogonal matrix, i.e., $V^{\top} V = I$
- Σ is a diagonal matrix of the form:

$$\begin{bmatrix} \sigma_1 & & & & 0 & 0 & 0 \\ & \ddots & & & & & \\ & & \sigma_i & & & & \\ & & & \ddots & & & \\ 0 & & & & \sigma_p & 0 & 0 \end{bmatrix}$$

$$p = \min\{m, n\}$$

$$\sigma_1 \geq \ldots \geq \sigma_i \geq \ldots \sigma_p \geq 0$$

  - σ_i are called singular values and they correspond to the eigenvalues of $A^{\top} A$.
  - $\rho(A)$ = number of nonzero singular values.

Spectral Theory

Let $\mathrm{A} \in \mathbb{R}^{n \times n}$

- $\mathrm{AA} = \mathrm{A}^2$
- $\mathrm{A}^4 = \mathrm{A} \cdot \mathrm{A} \cdot \mathrm{A} \cdot \mathrm{A}$
- $\mathrm{A}^0 = \mathrm{I}$
- A^{-1} is the inverse of A (if it exists).

$n \times n$ matrix of 0's

Theorem: (Cayley-Hamilton)

Every matrix satisfies its own characteristic equation, i.e., $\Delta(\mathrm{A}) = \mathbf{0}$

Spectral Theory

Example:

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

$\Delta(\lambda) = \lambda^2 - 5\lambda + 4$ (from previous example)

$$\Delta(A) = A^2 - 5A + 4I$$

Check if: $AA - 5A + 4I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Jordan Canonical form

<u>Theorem</u>: For any $n \times n$ matrix A there exists a non-singular matrix T such that

$$\mathrm{T}^{-1}\mathrm{A}\mathrm{T} = \begin{bmatrix} \boxed{L(\lambda_1)} & & & 0 \\ & \boxed{L(\lambda_2)} & & \\ & & \ddots & \\ 0 & & & \boxed{L(\lambda_n)} \end{bmatrix}$$

where $L(\lambda) = \begin{bmatrix} \lambda & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda \end{bmatrix}$

<u>Note</u>: If <u>all</u> eigenvalues are distinct then $L(\lambda_1) = \lambda_1$.

$$\mathrm{T}^{-1}\mathrm{A}\mathrm{T} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Jordan Canonical form (Examples)

Examples:

- $$T^{-1}AT = \left[\begin{array}{cc|cc} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ \hline 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{array}\right]$$

- $$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}$$

- $$\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$$

Normed Linear Spaces

Definition: Let $(\mathrm{X}, \mathcal{F})$ be a linear vector space. A real-valued function is called a **norm** (and is denoted by $\|.\|$) if the following properties hold:

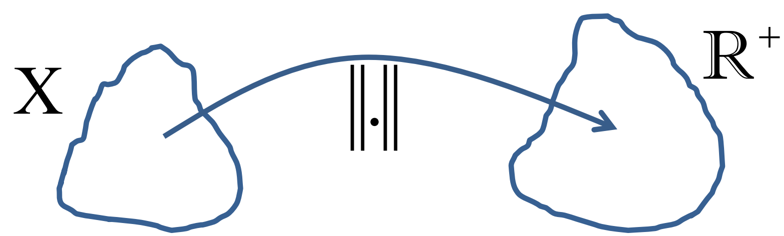
i. $\|x\| \geq 0$ and $\|x\| = 0 \Rightarrow x = 0_{\mathrm{X}} \quad \forall x \in \mathrm{X}.$

ii. $\|\alpha x\| = |\alpha| \|x\| \quad \forall x \in \mathrm{X}, \forall \alpha \in \mathcal{F}$

iii. $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in \mathrm{X}.$ (triangle inequality)

- A linear space with a norm is called a **normed linear space**.
- The norm can be considered as an operator.

$$\|.\| : \mathrm{X} \mapsto \mathbb{R}^+ \quad \left[\mathbb{R}^+ = \left\{x \in \mathbb{R} \,\middle|\, x \geq 0\right\}\right]$$



Normed Linear Spaces

Examples of norms:

a) $\mathrm{X} = \mathbb{R}^n$

absolute value

i. $\|x\|_1 := \sum_{i=1}^{n} |x_i|$ (1-norm)

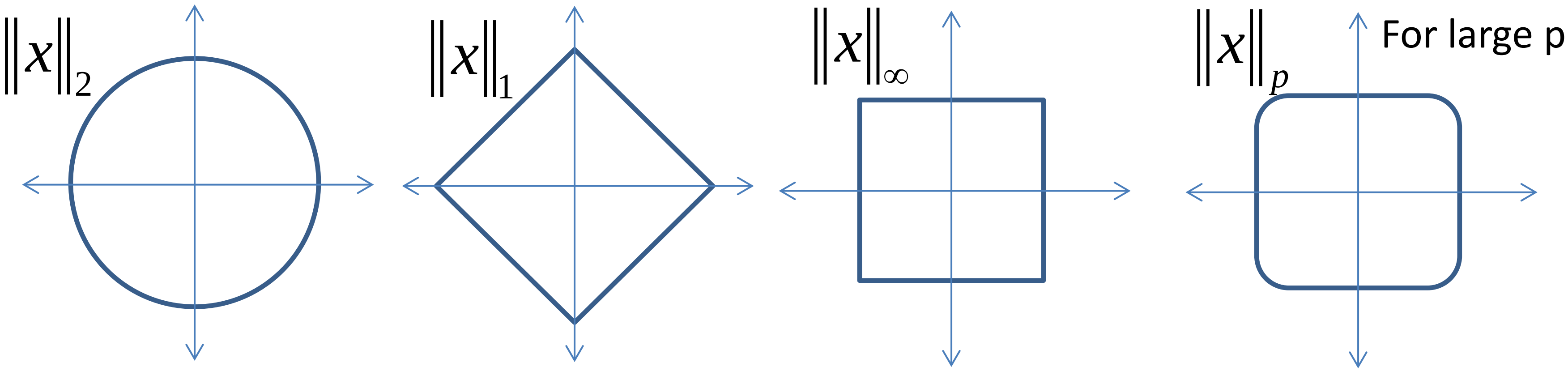
ii. $\|x\|_2 := \left[\sum_{i=1}^{n} |x_i|^2 \right]^{1/2}$ (2-norm / Euclidean norm)

iii. for $1 \le p < \infty$ $\|x\|_p := \left[\sum_{i=1}^{n} |x_i|^p \right]^{1/p}$ (p-norm)

iv. $\|x\|_\infty := \max_{1 \le i \le n} |x_i|$ (∞-norm)

Normed Linear Spaces

Consider the set: $U = \left\{ x \in \mathbb{R}^2 \;\middle|\; \|x\| = 1 \right\}$



Lemma: Let $\|\cdot\|$ and $\|\cdot\|'$ be any two norms on $\mathbb{R}^n$ (or $\mathbb{C}^n$). Then there exist constants $c_1, c_2 > 0$ so that

$$c_1 \|x\| \leq \|x\|' \leq c_2 \|x\| \quad \forall x \in \mathrm{X}.$$

Normed Linear Spaces

Notation: Normed linear spaces are sometimes denoted by

$$\left(\mathrm{X}, \mathcal{F}, \|.\|\right) \text{ or } \left(\mathrm{X}, \|.\|\right)$$

set field norm

For example: $\left(\mathbb{R}^5, \mathbb{R}, \|.\|_2\right)$, $\left(\mathbb{C}^3, \mathbb{C}, \|.\|_1\right)$

b) $\mathrm{X} = C\left[0, T\right]$ (can be extended to $C(-\infty, \infty)$)

i. $\|x\|_1 = \int_0^T \left|x(t)\right| \, dt$

ii. $\|x\|_2 = \left[\int_0^T \left|x(t)\right|^2 dt\right]^{1/2}$

iii. $\|x\|_p = \left[\int_0^T \left|x(t)\right|^p dt\right]^{1/p}$

iv. $\|x\|_\infty = \max_{t \in [0,T]} \left|x(t)\right|$

Normed Linear Spaces (Examples)

Consider $C\left[0,\infty\right]$ with

i. $x\left(t\right)=1$

$$\left\|x\right\|_{\infty}=\max_{t\in\left[0,\infty\right]}\left|1\right|=1$$

$$\left\|x\right\|_{1}=\int_{0}^{\infty}1=\infty$$

$$\left\|x\right\|_{p}=\infty \quad \forall p\in\left[1,\infty\right].$$

ii. $x\left(t\right)=e^{-t}$

$$\left\|x\right\|_{\infty}=\max_{t\in\left[0,\infty\right]}\left|e^{-t}\right|=1$$

$$\left\|x\right\|_{1}=\int_{0}^{\infty}e^{-t}=1$$

Normed Linear Spaces (Examples)

iii. $x(t) = \dfrac{1}{1+t}$

$$\|x\|_\infty = \max_{t\in[0,\infty]} \left| \frac{1}{1+t} \right| = 1$$

$\|x\|_2 < \infty$ but $\|x\|_1 = \infty$ (check!)

Operator Norms

Definition: A linear operator $\mathcal{A} : \mathrm{X} \mapsto \mathrm{Y}$ where X, Y are normed linear spaces, is said to be a **bounded linear operator** if there is a constant M such that

$$\left\|\mathcal{A}\left(x\right)\right\|_{\mathrm{Y}} \leq \mathrm{M}\left\|x\right\|_{\mathrm{X}} \quad \forall x \in \mathrm{X}$$

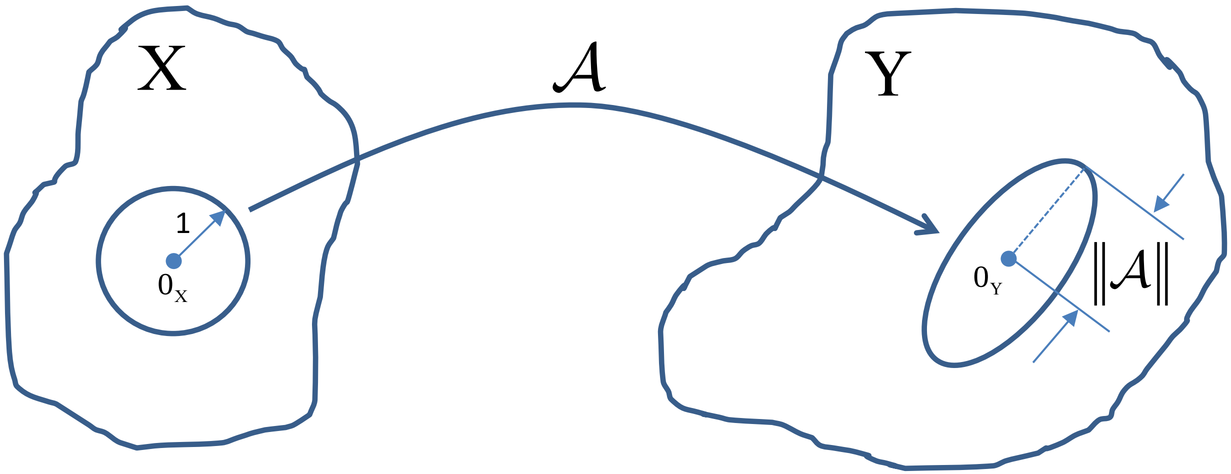
where M is independent of $x \in \mathrm{X}$.

The smallest such M that satisfies this condition is called the **norm of** $\mathcal{A}$ and is denoted by $\left\|\mathcal{A}\right\|$.

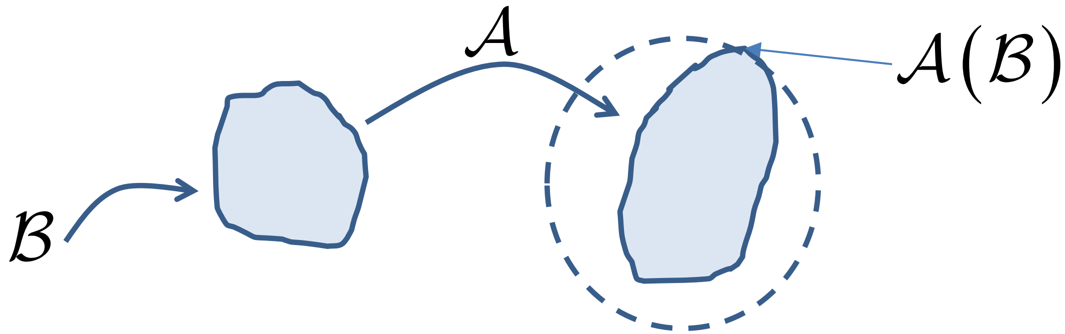
Note:

$$\begin{aligned}\left\|\mathcal{A}\right\| &= \max_{x \neq 0} \frac{\left\|\mathcal{A}\left(x\right)\right\|_{\mathrm{Y}}}{\left\|x\right\|_{\mathrm{X}}} \\ &= \max_{\left\|x\right\|_{\mathrm{X}}=1} \left\|\mathcal{A}\left(x\right)\right\|_{\mathrm{Y}} \\ &= \max_{\left\|x\right\|_{\mathrm{X}}\leq 1} \left\|\mathcal{A}\left(x\right)\right\|_{\mathrm{Y}}\end{aligned}$$

Geometry of an Operator Norm



OR:



Key questions:

- Does $\|\mathcal{A}\|$ satisfy the axioms of a norm?
- What is the linear space in this case?

Operator Norms

Lemma: Let $\mathcal{L}(\mathrm{X},\mathrm{Y})$ be the set of all linear transformations from X to Y; i.e.,

$$\mathcal{L}(\mathrm{X},\mathrm{Y}) := \left\{\mathcal{A} \mid \mathcal{A} : \mathrm{X} \mapsto \mathrm{Y}, \mathcal{A}\,\text{is linear}\right\}.$$

Then $\mathcal{L}(\mathrm{X},\mathrm{Y})$ is a linear space under the addition $\mathcal{A}+\mathcal{B}$ being defined as:

$$(\mathcal{A}+\mathcal{B})(x) = \mathcal{A}(x) + \mathcal{B}(x) \quad \mathcal{A},\mathcal{B} \in \mathcal{L}(\mathrm{X},\mathrm{Y}), x \in \mathrm{X}.$$

and scalar multiplication $\alpha\mathcal{A}$ defined as:

$$(\alpha\mathcal{A})(x) = \alpha\mathcal{A}(x) \quad \mathcal{A} \in \mathcal{L}(\mathrm{X},\mathrm{Y}), \alpha \in \mathcal{F}, x \in \mathrm{X}$$

Operator Norms

<u>Reminder:</u>

So far we have seen elements of linear spaces that are:

Vectors of reals $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ (e.g.), function (e.g., e^{-t})

Vectors of functions, $\begin{bmatrix} 5\sin(t) \\ \cos(t) \end{bmatrix}$, matrices, e.g. $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$,

Matrices of polynomials $\begin{bmatrix} 1+x & 1+x^3 \\ x^2+x^3 & 1+x^4 \end{bmatrix}$, and

now operators $\mathcal{A} : \mathrm{X} \mapsto \mathrm{Y}$

Operator Norms

Lemma: $\|\mathcal{A}\|$ is indeed a norm on $\mathcal{L}(\mathrm{X},\mathrm{Y})$

Proof: (i) obviously $\|\mathcal{A}\| \geq 0$ and $\|\mathcal{A}\| = 0 \Rightarrow \mathcal{A} = 0$

(ii) $$\|\alpha\mathcal{A}\| = \max_{\|x\|_{\mathrm{X}}=1} \|\alpha\mathcal{A}(x)\|_{\mathrm{Y}}$$

$$= \max_{\|x\|_{\mathrm{X}}=1} |\alpha| \|\mathcal{A}(x)\|_{\mathrm{Y}} = |\alpha| \|\mathcal{A}\|$$

(iii) $$\|\mathcal{A}+\mathcal{B}\| = \max_{\|x\|_{\mathrm{X}}=1} \|(\mathcal{A}+\mathcal{B})(x)\|_{\mathrm{Y}}$$

$$= \max_{\|x\|_{\mathrm{X}}=1} \|\mathcal{A}(x)+\mathcal{B}(x)\|_{\mathrm{Y}}$$

$$\leq \max_{\|x\|_{\mathrm{X}}=1} \left(\|\mathcal{A}(x)\|_{\mathrm{Y}} + \|\mathcal{B}(x)\|_{\mathrm{Y}} \right)$$

$$\leq \|\mathcal{A}\| + \|\mathcal{B}\|$$

Operator Norms

Examples of matrix norms:

Let $A \in \mathbb{R}^{n\times n}$ (considered as a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^n$)

$$\mathrm{A} = \left(\alpha_{ij}\right)$$

(i) $\|\mathrm{A}\|_\infty = \max_{\|x\|_\infty = 1} \|\mathrm{A}x\|_\infty$ $\quad \boxed{\|\mathrm{A}\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} \left|\alpha_{ij}\right|}$

(ii) $\|\mathrm{A}\|_1 = \max_{\|x\|_1 = 1} \|\mathrm{A}x\|_1$ $\quad \boxed{\|\mathrm{A}\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} \left|\alpha_{ij}\right|}$

(iii) $\|\mathrm{A}\|_2 = \max_{\|x\|_2 = 1} \|\mathrm{A}x\|_2$ $\quad \boxed{\|\mathrm{A}\|_2 = \left[\lambda_{\max}\left(\mathrm{A}^\top \mathrm{A}\right)\right]^{1/2}}$

($\lambda_{\max}\left(\mathrm{A}^\top \mathrm{A}\right)$ is the largest eigenvalue of $\mathrm{A}^\top \mathrm{A}$)

Example: $\mathrm{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad \|\mathrm{A}\|_1 = 6$; $\|\mathrm{A}\|_2 = 5.465$; $\|\mathrm{A}\|_\infty = 7$ (Check !)