Observers for nonlinear systems
Part I: Observability canonical form

The observability canonical form

In this Chapter we discuss the design of observers for nonlinear systems modeled by equations of the form

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \qquad (1)$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}$.

As it happens for the design of state-feedback stabilizing laws, where the design is facilitated if the equations that describe the system are put in special forms, also the design of observers is simplified if the equations describing the system are changed into equations having a special structure.

It is assumed that there exists a globally defined diffeomorphism

$$\begin{array}{cccc} \Phi : & \mathbb{R}^n & \to & \mathbb{R}^n \\ & x & \mapsto & z \end{array}$$

that carries system (1) into a system described by equations of the form

$$\begin{aligned} \dot{z}_1 &= \tilde{f}_1(z_1, z_2, u) \\ \dot{z}_2 &= \tilde{f}_2(z_1, z_2, z_3, u) \\ & \cdots \\ \dot{z}_{n-1} &= \tilde{f}_{n-1}(z_1, z_2, \ldots, z_n, u) \\ \dot{z}_n &= \tilde{f}_n(z_1, z_2, \ldots, z_n, u) \\ y &= \tilde{h}(z_1, u) \end{aligned} \qquad (2)$$

in which $\tilde{h}(z_1, u)$ and the $f_i(z_1, z_2, \ldots, z_{i+1}, u)$'s have the following properties

$$\frac{\partial \tilde{h}}{\partial z_1} \neq 0\,, \quad \text{and} \quad \frac{\partial \tilde{f}_i}{\partial z_{i+1}} \neq 0\,, \quad \text{for all} \quad i = 1, \ldots, n-1 \qquad (3)$$

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Note that, if the system were linear

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du\,, \end{aligned}$$

and observable, the change of variables

$$z = Tx = \begin{pmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{pmatrix} x\,.$$

would yield

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \cdots \\ \dot{z}_{n-1} \\ \dot{z}_n \end{pmatrix} = \begin{pmatrix} C \\ CA \\ \cdots \\ CA^{n-2} \\ CA^{n-1} \end{pmatrix} \dot{x} = \begin{pmatrix} CAx + CBu \\ CA^2x + CABu \\ \cdots \\ CA^{n-1}x + CA^{n-2}Bu \\ CA^nx + CA^{n-1}Bu \end{pmatrix} = \begin{pmatrix} z_2 + CBu \\ z_3 + CABu \\ \cdots \\ z_n + CA^{n-2}Bu \\ CA^nT^{-1}z + CA^{n-1}Bu \end{pmatrix}$$

$$y = z_1 + Du$$

which has the properties indicated above.

In general, suppose equations of the form (2), with properties (3), hold. Pick an input $u(\cdot)$, fix an initial condition $z(0)$ and let $z(t)$ and $y(t)$ denote the corresponding state and output trajectories.
Define a function $F_1(z_1, u, y)$ as

$$F_1(z_1, u, y) = y - \tilde{h}(z_1, u)\,.$$

By construction, the triplet $\{z_1(t), u(t), y(t)\}$ satisfies

$$F_1(z_1(t), u(t), y(t)) = 0\,. \tag{4}$$

The first condition in (3) implies

$$\frac{\partial F_1}{\partial z_1} \neq 0 \qquad \forall (z_1, u, y)\,.$$

Therefore, by the implicit function theorem, equation (4) can be solved for $z_1(t)$, at least in a neighborhood of a point $\{z_1(0), u(0), y(0)\}$ satisfying

$$y(0) = \tilde{h}(z_1(0), u(0))\,.$$

In other words, there is a map $z_1 = H_1(u, y)$ such that

$$F_1(H_1(u, y), u, y) = 0$$

for all (u, y) in a neighborhood of $(u(0), y(0))$ and $z_1(t)$ can be expressed as

$$z_1(t) = H_1(u(t), y(t))\,. \tag{5}$$

In summary, the value of z_1 at time t can be expressed as an explicit function of the values of u and y at the same time t.

Next, consider a function $F_2(z_2, z_1, \dot{z}_1, u)$ defined as

$$F_2(z_2, z_1, \dot{z}_1, u) = \dot{z}_1 - \tilde{f}_1(z_1, z_2, u)\,.$$

By construction, the triplet $\{z_1(t), z_2(t), u(t)\}$ satisfies

$$F_2(z_2(t), z_1(t), \dot{z}_1(t), u(t)) = 0\,. \tag{6}$$

The second condition in (3) implies

$$\frac{\partial F_2}{\partial z_2} \neq 0 \qquad \forall (z_2, z_1, \dot{z}_1, u)\,.$$

Therefore equation (6) can be solved for $z_2(t)$, at least in a neighborhood of a point $(z_2(0), z_1(0), \dot{z}_1(0), u(0))$ satisfying

$$\dot{z}_1(0) = \tilde{f}_1(z_1(0), z_2(0), u(0))\,.$$

In other words, there is a map $z_2 = H_2(z_1, \dot{z}_1, u)$ such that

$$F_2(H_2(z_1, \dot{z}_1, u), z_1, \dot{z}_1, u) = 0$$

for all $(z_1, \dot{z}_1, u)$ in a neighborhood of $(z_1(0), \dot{z}_1(0), u(0))$ and $z_2(t)$ can be expressed as

$$z_2(t) = H_2(z_1(t), \dot{z}_1(t), u(t))\,. \tag{7}$$

In view of the previous calculation, which has shown that $z_1(t)$ can be expressed as in (5), define a function $\dot{H}_1(u, u^{(1)}, y, y^{(1)})$ as

$$\dot{H}_1(u, u^{(1)}, y, y^{(1)}) = \frac{\partial H_1}{\partial u} u^{(1)} + \frac{\partial H_1}{\partial y} y^{(1)}$$

and observe that

$$\dot{z}_1(t) = \dot{H}_1(u(t), u^{(1)}(t), y(t), y^{(1)}(t)) .$$

Entering this in (7), it is seen that $z_2(t)$ can be expressed as

$$z_2(t) = H_2(H_1(u(t), y(t)), \dot{H}_1(u(t), u^{(1)}(t), y(t), y^{(1)}(t)), u(t)) .$$

In summary, the value of z_2 at time t can be expressed as an explicit function of the values of $u, u^{(1)}$ and $y, y^{(1)}$ at the same time t.

Proceeding in this way and using all of the (3), it can be concluded that, at least locally around a point $z(0), u(0), y(0)$, all the components $z_i(t)$ of $z(t)$ can be explicitly expressed as functions of the values $u(t), u^{(1)}(t), \ldots, u^{(i-1)}(t)$ and $y(t), y^{(1)}(t), \ldots, y^{(i-1)}(t)$ of input, output and their higher-order derivatives with respect to time.

In other words, the values of input, output and their derivatives with respect to time up to order $n-1$, at any given time t, uniquely determine the value of the state at this time t.

Existence of observability canonical form

Consider again system (1), suppose that $f(0,0)=0$, $h(0,0)=0$, and define – recursively – a sequence of real-valued functions $\varphi_i(x,u)$ as follows

$$\varphi_1(x,u) := h(x,u)\,, \qquad \varphi_i(x,u) := \frac{\partial \varphi_{i-1}}{\partial x} f(x,u)\,, \tag{8}$$

for $i = 2, \ldots, n$.

Using such functions, define a sequence of $\mathbb{R}^i$-valued functions $\Phi_i(x,u)$ as follows

$$\Phi_i(x,u) = \begin{pmatrix} \varphi_1(x,u) \\ \vdots \\ \varphi_i(x,u) \end{pmatrix}$$

for $i = 1, \ldots, n$.

Finally, with each of such $\Phi_i(x,u)$'s, associate the subspace

$$\mathcal{K}_i(x,u) = \mathrm{Ker}\left[\frac{\partial \Phi_i}{\partial x}\right]_{(x,u)}.$$

The collection of all such subspaces is called the canonical flag of (1). The canonical flag is said to be uniform if

(i) for all $i = 1, \ldots, n$, for all $u \in \mathbb{R}^m$ and for all $x \in \mathbb{R}^n$

$$\mathrm{dim}\mathcal{K}_i(x,u) = n - i\,.$$

(ii) for all $i = 1, \ldots, n$ and for all $x \in \mathbb{R}^n$

$$\mathcal{K}_i(x,u) = \text{independent of } u\,.$$

In other words, condition (i) says that the subspace $\mathcal{K}_i(x,u)$ has constant dimension $n-i$ for all (x,u). Condition (ii) says that, for each x, the subspace $\mathcal{K}_i(x,u)$ is always the same, regardless of what $u \in \mathbb{R}^m$ is.

Proposition System (1) is globally diffeomorphic to a system in Gauthier-Kupca's observability canonical form only if its canonical flag is uniform.

Proposition Consider the nonlinear system (1) and define a map

$$\begin{array}{rcccl} \Phi & : & \mathbb{R}^n & \to & \mathbb{R}^n \\ & & x & \mapsto & z = \Phi(x) \end{array}$$

as

$$\Phi(x) = \begin{pmatrix} \varphi_1(x,0) \\ \varphi_2(x,0) \\ \cdots \\ \varphi_n(x,0) \end{pmatrix}.$$

Suppose that:

(i) the canonical flag of (1) is uniform,

(ii) $\Phi(x)$ is a global diffeomorphism.

Then, system (1) is globally diffeomorphic, via $z = \Phi(x)$, to a system in Gauthier-Kupca's uniform observability canonical form.

The case of input-affine systems

If the system is input-affine, i.e. modeled by equations of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x)\,. \end{aligned} \tag{9}$$

then

$$\Phi_n(x,0) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \cdots \\ L_f^{n-1} h(x) \end{pmatrix} := \Phi(x).$$

The canonical form is

$$\begin{aligned} \dot{z}_1 &= z_2 + \tilde{g}_1(z_1)u \\ \dot{z}_2 &= z_3 + \tilde{g}_2(z_1,z_2)u \\ &\cdots \\ \dot{z}_{n-1} &= z_n + \tilde{g}_{n-1}(z_1,z_2,\ldots,z_{n-1})u \\ \dot{z}_n &= \tilde{f}_n(z_1,z_2,\ldots,z_n) + \tilde{g}_n(z_1,z_2,\ldots,z_n)u \\ y &= z_1 \end{aligned}$$