Stabilization by Output Feedback
Part II: A robust separation principle

Consider a single-input single-output system having relative degree $r > 1$, and possessing a globally defined normal form, which can be written as

$$\begin{aligned} \dot{z} &= f(z,\xi) \\ \dot{\xi} &= \hat{A}\xi + \hat{B}[q(z,\xi) + b(z,\xi)u] \\ y &= \hat{C}\xi\,, \end{aligned} \tag{1}$$

where $z \in \mathbb{R}^{n-r}$, $\xi \in \mathbb{R}^r$ and

$$\hat{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Assume that

$$q(0,0) = 0 \tag{2}$$

and that the coefficient $b(z,\xi)$ satisfies

$$0 < b_{\min} \le b(z,\xi) \le b_{\max} \qquad \text{for all } (z,\xi) \tag{3}$$

for some $b_{\min}, b_{\max}$.
Finally, assume that $f(0,0) = 0$

$$\dot{z} = f(z,\xi)\,, \tag{4}$$

viewed as a system with input ξ and state z, is input-to-state stable. That is, assume that system (1) is strongly minimum phase.

We have seen that the feedback law

$$u = \frac{1}{b(z,\xi)}\left(K\xi - q(z,\xi)\right), \tag{5}$$

if K is such that $(\hat{A} + \hat{B}K)$ is a Hurwitz matrix, globally asymptotically stabilizes the equilibrium $(z,\xi) = (0,0)$ of the resulting closed-loop system.

However, the implementation of this law requires accurate knowledge of $b(z,\xi)$ and $q(z,\xi)$ and availability of the full state (z,ξ).

We will see in what follows that a suitable "asymptotic proxy" of this law can be designed, which does not suffer such limitations.

The idea is to use the measured output y to drive an appropriate dynamical system to the purpose of estimating the components of ξ as well as to overcome the necessity of knowing the functions $b(z,\xi)$ and $q(z,\xi)$.

To this end, let $\psi(\xi,\sigma)$ be the function defined as

$$\psi(\xi,\sigma) = \frac{1}{b_0}[K\xi - \sigma],$$

in which $\xi \in \mathbb{R}^r$, $\sigma \in \mathbb{R}$, b_0 is a design parameter and K a vector with the properties indicated above (i.e. such that $(\hat{A} + \hat{B}K)$ is a Hurwitz matrix).

Moreover, let $g : \mathbb{R} \to \mathbb{R}$ be a smooth "saturation" function, characterized as follows: $g(s) = s$ if $|s| \leq \ell$, $g(s)$ is odd and monotonically increasing, with $0 < g'(s) \leq 1$, and $\lim_{s\to\infty} g(s) = L(1 + c)$ with $0 < c \ll 1$.

The "saturation level" ℓ, is a design parameter that will be determined later.

System (1) is controlled by a feedback law of the form

$$u = g(\psi(\hat{\xi}, \sigma))\,, \tag{6}$$

in which $\hat{\xi} \in \mathbb{R}^r$ and σ are states of the dynamical system

$$\begin{aligned}
\dot{\hat{\xi}}_1 &= \hat{\xi}_2 + \kappa\alpha_1(y - \hat{\xi}_1)\\
\dot{\hat{\xi}}_2 &= \hat{\xi}_3 + \kappa^2\alpha_2(y - \hat{\xi}_1)\\
&\cdots\\
\dot{\hat{\xi}}_{r-1} &= \hat{\xi}_r + \kappa^{r-1}\alpha_{r-1}(y - \hat{\xi}_1)\\
\dot{\hat{\xi}}_r &= \sigma + b_0 g(\psi(\hat{\xi}, \sigma)) + \kappa^r\alpha_r(y - \hat{\xi}_1)\\
\dot{\sigma} &= \kappa^{r+1}\alpha_{r+1}(y - \hat{\xi}_1)\,.
\end{aligned} \tag{7}$$

The coefficients κ and $\alpha_1, \alpha_2, \ldots, \alpha_{r+1}$ are design parameters.
The dynamical system thus defined has the typical structure of an "observer". In the analysis of the asymptotic properties of the resulting closed-loop system, it is convenient to replace $\hat{\xi}_1, \ldots, \hat{\xi}_r, \sigma$ by means of (scaled) "error" variables, defined as follows

$$\begin{aligned}
e_1 &= \kappa^r(\xi_1 - \hat{\xi}_1)\\
e_2 &= \kappa^{r-1}(\xi_2 - \hat{\xi}_2)\\
&\cdots\\
e_r &= \kappa(\xi_r - \hat{\xi}_r)\\
e_{r+1} &= q(z, \xi) + [b(z, \xi) - b_0]g(\psi(\xi, \sigma)) - \sigma\,.
\end{aligned} \tag{8}$$

The first r of these relations can be trivially inverted, to recover each $\hat{\xi}_i$, as function of e_i and ξ_i.

To recover σ form the latter, b_0 needs to be chosen appropriately. To this end, bearing in mind the expression of $\psi(\xi,\sigma)$, observe that the relation in question is equivalent to the following one

$$\frac{K\xi - q(z,\xi) + e_{r+1}}{b(z,\xi)} = \frac{b_0}{b(z,\xi)}\Big[\Big(\frac{b(z,\xi)-b_0}{b_0}\Big)g(\psi(\xi,\sigma)) + \psi(\xi,\sigma)\Big]. \qquad (9)$$

If one sets

$$\psi^*(z,\xi,e_{r+1}) = \frac{K\xi - q(z,\xi) + e_{r+1}}{b(z,\xi)}$$

and defines a function $F : \mathbb{R} \to \mathbb{R}$ as

$$F(s) = \frac{b_0}{b(z,\xi)}\Big[\Big(\frac{b(z,\xi)-b_0}{b_0}\Big)g(s) + s\Big] \qquad (10)$$

the relation (9) can be simply rewritten as

$$\psi^* = F(\psi)\,.$$

Since $b(z,\xi)$, by assumption, is bounded as in (3), it is always possible to pick b_0 so as to make

$$\Big|\frac{b(z,\xi)-b_0}{b_0}\Big| \le \delta_0 < 1 \qquad \text{for some } \delta_0. \qquad (11)$$

Thus, since $0 < g'(s) \le 1$ by hypothesis, if b_0 is chosen in this way, $F'(s)$ is strictly positive, i.e. $F(s)$ is a strictly increasing (odd) function.
Moreover, since $\lim_{s\to\infty} g(s) = L(1+c)$, it is seen that $\lim_{s\to\infty} F(s) = \infty$, and consequently $F(\mathbb{R}) = \mathbb{R}$.
In summary, $F(s)$ is globally invertible.

It is also worth noting that, so long as $|s| \leq L$, the function $F(s)$ is an identity, i.e. $F(s) = s$.

Hence, if b_0 is chosen to satisfy (11), one has

$$\psi = F^{-1}(\psi^*)$$

and this – bearing in mind the expressions of ψ and ψ^* – shows that σ can always be recovered, from the last of (8), as a smooth function of z, ξ, e_{r+1}. That is, $\hat{\xi} = \hat{\xi}(\xi, e)$ and $\sigma = \sigma(z, \xi, e)$.

Appropriate calculations show that the variable $e = \mathrm{col}(e_1, \ldots, e_{r+1})$ thus defined satisfies an equation having the following structure

$$\dot{e} = \kappa[A - BC\Delta_0(z, \xi, e)]e + B_1\Delta_1(z, \xi, e) + B_2\Delta_2(z, \xi, e)\,, \qquad (12)$$

in which

$$A = \begin{pmatrix} -\alpha_1 & 1 & 0 & \cdots & 0 \\ -\alpha_2 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & 0 \\ -\alpha_r & \cdot & \cdot & \cdots & 1 \\ -\alpha_{r+1} & \cdot & \cdot & \cdots & 0 \end{pmatrix}, \qquad B = B_2 = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 1 \\ 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \alpha_{r+1} & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Thus, in summary, the controlled system is viewed as interconnection of the form

$$\begin{aligned} \dot{z} &= f(z,\xi) \\ \dot{\xi} &= (\hat{A}+\hat{B}K)\xi + \hat{B}[q(z,\xi)+b(z,\xi)g(\tfrac{K\hat{\xi}-\sigma}{b_0}) - K\xi]\,, \end{aligned} \tag{13}$$

in which $\hat{\xi}$ and σ are to be seen as functions of e, namely $\hat{\xi} = \hat{\xi}(\xi, e)$ and $\sigma = \sigma(z, \xi, e)$,

$$\dot{e} = \kappa[A - BC\Delta_0(z,\xi,e)]e + B_1\Delta_1(z,\xi,e) + B_2\Delta_2(z,\xi,e) \tag{14}$$

The "coupling terms" $\Delta_0(z,\xi,e), \Delta_1(z,\xi,e), \Delta_2(z,\xi,e)$ are suitable real-valued functions having the following properties.

If b_0 is chosen so as to satisfy (11), then $|\Delta_0(z,\xi,e)| \leq \delta_0 < 1$, for some δ_0. $|\Delta_1(z,\xi,e)| < \delta_1|e|$ for some δ_1. For any compact set $\mathcal{S}$ there is a number $M_{\mathcal{S}}$ such that $|\Delta_2(z,\xi,e)| \leq M_{\mathcal{S}}$ for all $(z,\xi) \in \mathcal{S}$ and all $e \in \mathbb{R}^{r+1}$. This number $M_{\mathcal{S}}$ is independent of κ.

If the design parameter κ is large, the system (14)–(**??**) has the standard form of a two-time-scale system.

Lemma There exist a choice of the coefficients $\alpha_0, \cdots, \alpha_r$, a positive definite and symmetric $(r+1) \times (r+1)$ matrix P and a number $\lambda > 0$ such that

$$P[A - BC\Delta_0(z, \xi, e)] + [A - BC\Delta_0(z, \xi, e)]^\top P \leq -\lambda I. \qquad (15)$$

Lemma Let the α_i's be chosen so as to make (15) satisfied. Suppose $(z(t), \xi(t)) \in \mathcal{S}$ for all $t \in [0, T_{\max})$ and suppose that $\|(\hat{\xi}, \sigma)(0)\| \leq R$. Then, for every $0 < T \leq T_{\max}$ and every $\varepsilon > 0$, there is a κ^* such that, for all $\kappa \geq \kappa^*$,

$$\|e(t)\| \leq 2\varepsilon \quad \textit{for all } t \in [T, T_{\max}).$$

Proposition Consider system (1), controlled by (6)–(7). Suppose that (2) holds and that $b(x, \xi)$ is bounded as in (3). Suppose (1) is strongly minimum phase (with respect to the set $\mathcal{A} = \{0\}$). Let K be such that $\hat{A} + \hat{B}K$ is Hurwitz. For every choice of a compact set $\mathcal{C}$, there is a choice of the design parameters b_0, L and $\alpha_1, \ldots, \alpha_{r+1}$ and a number κ^* such that, for all $\kappa \geq \kappa^*$, the equilibrium $(z, \xi, \hat{\xi}, \sigma) = (0, 0, 0, 0)$ is asymptotically stable, with a domain of attraction that contains the set $\mathcal{C}$.

Asymptotic performance recovery

It can be shown that, for $t \geq T$ (where T can be made arbitrarily small), the upper system of the interconnection becomes

$$\begin{aligned}\dot{z} &= f(z,\xi)\\ \dot{\xi} &= (\hat{A}+\hat{B}K)\xi + \mathcal{O}(e(t))\,,\end{aligned}$$

in which $e(t)$ asymptotically decays to zero.

Thus, the behavior of the system asymptotically converges to the behavior of the feedback linearized system

$$\begin{aligned}\dot{z} &= f(z,\xi)\\ \dot{\xi} &= (\hat{A}+\hat{B}K)\xi\,,\end{aligned}$$

In other words, the proposed robust control law asymptotically recovers the performance achieved by means of the non-robust feedback linearizing law.