Stabilization of Nonlinear Systems via State Feedback
Part II: Stabilization via Full and Partial State

Stabilization via full state feedback

If a system is strongly minimum-phase, it is quite easy to design a globally stabilizing state feedback law.

Consider a system in normal form, which we assume to be globally defined, and assume that the system is strongly minimum-phase, i.e. assume that $f_0(0,0) = 0$ and that

$$\dot{z} = f_0(z,\xi)\,,$$

viewed as a system with input ξ and state z, is input-to-state stable.

Since the coefficient $b(z,\xi)$ is nowhere zero, consider the feedback law

$$u = \frac{1}{b(z,\xi)}\,(-q(z,\xi) + \hat{K}\xi)\,, \tag{1}$$

in which $\hat{K} \in \mathbb{R} \times \mathbb{R}^r$ is a vector of design parameters.

Under such feedback law, the system becomes

$$\begin{aligned} \dot{z} &= f_0(z,\xi) \\ \dot{\xi} &= (\hat{A} + \hat{B}\hat{K})\xi\,. \end{aligned} \tag{2}$$

Since the pair $(\hat{A},\hat{B})$ is reachable, it is possible to pick $\hat{K}$ so that the matrix $(\hat{A} + \hat{B}\hat{K})$ is a Hurwitz matrix. If this is the case, system (2) appears as a cascade-connection in which a globally asymptotically stable system (the lower sub-system) drives an input-to-state stable system (the upper sub-system).

According a standard result, such cascade-connection is globally asymptotically stable.

The feedback law (1) is expressed in the (z, ξ) coordinates that characterize the normal form. To express it in the original coordinates that characterize the model of the system, it suffices to bear in mind that

$$b(z,\xi) = L_g L_f^{r-1} h(x)\,, \qquad q(z,\xi) = L_f^r h(x)$$

and to observe that, since $\xi_i = L_f^{i-1} h(x)$ for $i = 1, \ldots, r$, then

$$\hat{K}\xi = \sum_{i=1}^{r} \hat{k}_i L_f^{i-1} h(x)$$

in which $\hat{k}_1, \ldots, \hat{k}_r$ are the entries of the row vector $\hat{K}$. Thus, we can conclude what follows.

Proposition Consider a system of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{3}$$

with $f(0) = 0$ and $h(0) = 0$. Suppose the system has relative degree r and possesses a globally defined normal form. Suppose the system is strongly minimum-phase. If $\hat{K} \in \mathbb{R} \times \mathbb{R}^r$ is any vector such that $\sigma(\hat{A} + \hat{B}\hat{K}) \in \mathbb{C}^-$, the state feedback law

$$u(x) = \frac{1}{L_g L_f^{r-1} h(x)} \left(-L_f^r h(x) + \sum_{i=1}^{r} \hat{k}_i L_f^{i-1} h(x) \right), \tag{4}$$

globally asymptotically stabilizes the equilibrium $x = 0$.

This design method is also known as feedback linearization. In fact, the dynamics of the second set of equations, that is the only dynamics that affects the output, is that of a linear system.

This feedback strategy, although very intuitive and elementary, is not useful in a practical context because it relies upon exact cancelation of certain nonlinear function and, as such, possibly non-robust.

Uncertainties in $q(z,\xi)$ and $b(z,\xi)$ would make this strategy un-applicable.

Moreover, the implementation of such control law requires the availability, for feedback purposes, of the full state (z,ξ) of the system, a condition that might be hard to ensure.

Motivated by these considerations, we will re-address the problem by seeking feedback laws depending on fewer measurements (hopefully only on the measured output y) and possibly robust with respect to model uncertainties. Of course, in return, some price has to be paid.

The assumption that the system is strongly minimum-phase can be weakened if a system has the special structure

$$
\begin{array}{rcl}
\dot{z} & = & f_0(z, \xi_1) \\
\dot{\xi}_1 & = & \xi_2 \\
 & & \cdots \\
\dot{\xi}_{r_1} & = & \xi_r \\
\dot{\xi}_r & = & q(z, xi) + b(z, \xi)u \,.
\end{array}
\tag{5}
$$

The design of a stabilizing feedback law is based on a recursive procedure, known as backstepping, by means of which it is possible to construct, for a system having such special structure, a state feedback stabilizing law as well as a Lyapunov function.

The procedure in question reposes on the following results.

Lemma Consider a system described by equations of the form

$$\begin{aligned} \dot z &= f(z,\xi) \\ \dot\xi &= u \end{aligned} \tag{6}$$

in which $(z,\xi) \in \mathbb{R}^{n-r} \times \mathbb{R}$, and $f(0,0)=0$. Suppose the equilibrium $z=0$ of $\dot z = f(z,0)$ is globally asymptotically stable, with Lyapunov function $V(z)$. Express $f(z,\xi)$ in the form [1]

$$f(z,\xi) = f(z,0) + p(z,\xi)\xi \tag{7}$$

where $p(z,\xi)$ is a smooth function.
Set

$$u(z,\xi) = -\xi - \frac{\partial V}{\partial z} p(z,\xi)\,. \tag{8}$$

Then, the equilibrium $(z,\xi)=(0,0)$ of (10) controlled by (8) is globally asymptotically stable, with Lyapunov function

$$W(z,\xi) = V(z) + \frac{1}{2}\xi^2\,.$$

[1] To check that this is always possible, observe that the difference

$$\bar f(z,\xi) = f(z,\xi) - f(z,0)$$

is a smooth function vanishing at $\xi = 0$, and express $\bar f(z,\xi)$ as

$$\bar f(z,\xi) = \int_0^1 \frac{\partial \bar f(z,s\xi)}{\partial s} ds = \int_0^1 \left[\frac{\partial \bar f(z,\zeta)}{\partial \zeta}\right]_{\zeta = s\xi} \xi ds\ .$$

Proof. By assumption, $V(z)$ is positive definite and proper, which implies that the function $W(z,\xi)$ in the Lemma is positive definite and proper as well. Moreover,

$$\frac{\partial V}{\partial z} f(z,0) \le -\alpha(\|z\|) \qquad \forall z \in \mathbb{R}^{n-r}$$

for some class $\mathcal{K}$ function $\alpha(\cdot)$. Observe that

$$\dot{W} = \frac{\partial W}{\partial z} f(z,\xi) + \frac{\partial W}{\partial \xi} u = \frac{\partial V}{\partial z} f(z,0) + \frac{\partial V}{\partial z} p(z,\xi)\xi + \xi u \,.$$

Choosing u as in (8) yields

$$\dot{W} \le -\alpha(\|z\|) - \xi^2 \qquad \forall (z,\xi) \in \mathbb{R}^{n-r} \times \mathbb{R} \,.$$

The quantity on the right-hand side is negative for all nonzero (z,ξ) and this proves the Lemma.

This result can extended by showing that, to the purpose of stabilizing the equilibrium $(z,\xi)=(0,0)$ of system (10), it suffices to assume that the equilibrium $z=0$ of

$$\dot{z}=f(z,\xi)\,,$$

viewed as a system with state z and input ξ, is stabilizable by means of a smooth control law $\xi=\xi^*(z)$.

To see that this is the case, change the variable ξ of (10) into

$$\zeta=\xi-\xi^*(z)\,,$$

which transforms (10) into a system

$$\begin{aligned}\dot{z}&=f(z,\xi^*(z)+\zeta)\\ \dot{\zeta}&=-\frac{\partial\xi^*}{\partial z}f(z,\xi^*(z)+\zeta)+u\,.\end{aligned}\tag{9}$$

Pick now

$$u=-\frac{\partial\xi^*}{\partial z}f(z,\xi^*(z)+\zeta)+\bar{u}$$

so as to obtain a system of the form

$$\begin{aligned}\dot{z}&=f(z,\xi^*(z)+\zeta)\\ \dot{\zeta}&=\bar{u}\,.\end{aligned}$$

This system has the same structure as that of system (10), and – by construction – satisfies the assumptions of the Lemma.

Thus, this system can be globally stabilized by means of a control $\bar{u}$ having the structure of the control indicated in that Lemma.

The function $\xi^*(z)$, which is seen as a "control" imposed on the upper subsystem of (10), is usually called a virtual control.

Lemma Consider a system described by equations of the form

$$\begin{aligned} \dot{z} &= f(z,\xi) \\ \dot{\xi} &= u \end{aligned} \tag{10}$$

in which $(z,\xi) \in \mathbb{R}^{n-r} \times \mathbb{R}$, and $f(0,0) = 0$. Suppose the equilibrium $z = 0$ of $\dot{z} = f(z,\xi)$ is globally asymptotically stabilized by means of a virtual control $\xi^*(z)$.

Then, the system can be globally asymptotically stabilized by means of a control $u(z,\xi)$.

The property indicated above can be used repeatedly, to address the problem of stabilizing a system of the form (5).

In the first iteration, beginning from a virtual control $\xi_1^*(z)$ that stabilizes the equilibrium $z = 0$ of

$$\dot{z} = f_0(z, \xi_1^*(z)),$$

one finds a virtual control $\xi_2^*(z, \xi_1)$ that stabilizes the equilibrium $(z, \xi_1) = (0, 0)$ of

$$\begin{array}{rcl} \dot{z} & = & f_0(z, \xi_1) \\ \dot{\xi}_1 & = & \xi_2^*(z, \xi_1). \end{array}$$

Then, using the Lemma again, one finds a virtual control $\xi_3^*(z, \xi_1, \xi_2)$ that stabilizes the equilibrium $(z, \xi_1, \xi_2) = (0, 0, 0)$ of

$$\begin{array}{rcl} \dot{z} & = & f_0(z, \xi_1) \\ \dot{\xi}_1 & = & \xi_2 \\ \dot{\xi}_2 & = & \xi_3^*(z, \xi_1). \end{array}$$

and so on.

Note that, while the feedback law (1) has a very simple expression, the feedback law derived above cannot be easily expressed in closed form.

Rather, it can only be derived by means of a recursive procedure.

The actual expression of the law in question also requires the explicit knowledge of the function $V(z)$.

In return, the stabilization method just described does not require the upper sub-system of the form (5) to be input-to-state stable, as assumed in the case of the law (1), but only relies upon the assumption that the sub-system in question is stabilizable, by means of an appropriate virtual control $\xi_1^*(z)$.

Note also that the method in question requires availability of the full state and is not robust, as it reposes on exact cancelations.

We address now the issue of using only a partial state information to stabilize the system.

We consider first the case of a system having relative degree 1, which in normal form is written as

$$\begin{aligned} \dot{z} &= f_0(z,\xi) \\ \dot{\xi} &= q(z,\xi) + b(z,\xi)u \\ y &= \xi \end{aligned} \tag{11}$$

in which $z \in \mathbb{R}^{n-1}$ and $\xi \in \mathbb{R}$. As before, we assume that

$$\begin{aligned} f_0(0,0) &= 0 \\ q(0,0) &= 0\,. \end{aligned}$$

The coefficient $b(z,\xi)$ is by definition nowhere zero. Being a continuous function of (z,ξ), it is either always positive or always negative. Witout loss of generality, we assume

$$b(z,\xi) > 0 \qquad \text{for all } (z,\xi)\,.$$

We retain the assumption that the system is globally minimum-phase.

The system is controlled by the very simple feedback law

$$u = -ky \tag{12}$$

with $k > 0$, which yields a closed-loop system

$$\begin{aligned} \dot{z} &= f_0(z, \xi) \\ \dot{\xi} &= q(z, \xi) - b(z, \xi)k\xi\,. \end{aligned} \tag{13}$$

Set

$$x = \mathrm{col}(z, \xi)$$

and rewrite (13) as

$$\dot{x} = F_k(x) \tag{14}$$

in which

$$F_k(x) = \begin{pmatrix} f_0(z, \xi) \\ q(z, \xi) - b(z, \xi)k\xi \end{pmatrix}.$$

Proposition Consider system (3), with $f(0) = 0$ and $h(0) = 0$. Suppose the system has relative degree 1 and possesses a globally defined normal form. Suppose the system is globally minimum-phase. Let the control be provided by the output feedback $u = -ky$ so that a closed-loop system modeled as in (14) is obtained. Then, for every choice of a compact set $\mathcal{C}$ and of a number $\varepsilon > 0$, there is a number k^* and a finite time T such that, if $k \geq k^*$, all trajectories of the closed-loop system with initial condition $x(0) \in \mathcal{C}$ remain bounded and satisfy $\|x(t)\| < \varepsilon$ for all $t \geq T$.

This Proposition shows that, no matter how large the set $\mathcal{C}$ of initial conditions is chosen and no matter how small a "target set" B_ϵ is chosen, there is a value k^* of the gain in (12) and a finite time T such that, if the actual gain parameter k is larger than or equal to k^*, all trajectories of the closed-loop system with origin in $\mathcal{C}$ are bounded and, for all $t \geq T$, remain in the set B_ϵ.

This property is commonly referred to by saying that the control law (12) is able to semi-globally and practically stabilize the point $(z, \xi) = (0, 0)$.

The term "practical" (as opposite to asymptotic) is meant to stress the fact that the convergence is not to a point, but rather to a neighborhood of that point, that can be chosen arbitrarily small, while the term "semiglobal" (as opposite to global) is meant to stress the fact that the convergence to the target set is not for all initial conditions, but rather for a compact set of initial conditions, that can be chosen arbitrarily large.

The standing assumption here is that the system is globally minimum-phase (as opposite to strongly minimum phase) and the stability result is obtained via high-gain output feedback.

The minimal value k^* of the feedback gain k is determined by the choice of the set $\mathcal{C}$ and by the value of ϵ. In particular, as it appears from the proof, k^* increases as $\mathcal{C}$ "increases" (in the sense of set inclusion) and also increases as ϵ decreases.

To obtain asymptotic stability, either a nonlinear control law $u = -\kappa(y)$ is needed or, if one insists in using a linear law $u = -ky$, the extra assumption that the system is also locally exponentially minimum phase, is necessary.

Consider, the candidate Lyapunov function

$$W(x) = V(z) + \frac{1}{2}\xi^2$$

For any real number $a > 0$, let Ω_a denote the sublevel set of $W(x)$

$$\Omega_a = \{x \in \mathbb{R}^n : W(x) \le a\}$$

and let $B_\epsilon = \{x \in \mathbb{R}^n : \|x\| < \epsilon\}$ denote the (open) ball radius ϵ. Assume, without loss of generality, that $\mathcal{C}$ is such that $B_\epsilon \subset \mathcal{C}$.

Since $W(x)$ is positive definite and proper, there exist numbers $0 < d < c$ such that

$$\Omega_d \subset B_\epsilon \subset \mathcal{C} \subset \Omega_c\,.$$

Consider also the compact "annular" region

$$S_d^c = \{x \in \mathbb{R}^n : d \le W(x) \le c\}\,.$$

It will be shown that – if the gain coefficient k is large enough – the function

$$\dot{W}(x) := \frac{\partial W}{\partial x} F_k(x) = \frac{\partial V}{\partial z} f_0(z,\xi) + \xi q(z,\xi) - b(z,\xi) k \xi^2$$

is *negative* at each point of S_d^c.

To this end, proceed as follows. Consider the compact set

$$S_0 = \{x \in S_d^c : \xi = 0\}\,.$$

At each point of S_0

$$\dot{W}(x) = \frac{\partial V}{\partial x} f_0(z,0) \le -\alpha(\|z\|)$$

Since $\min_{x\in S_0} \|z\| > 0$, there is a number $a > 0$ such that

$$\dot{W}(x) \le -a \qquad \forall x \in S_0\,.$$

Hence, by continuity, there is an open set $S' \supset S_0$ such

$$\dot{W}(x) \le -a/2 \qquad \forall x \in S'\,. \tag{15}$$

Consider now the set

$$S'' = \{x \in S_d^c : x \notin S'\}\,.$$

which is a compact set (and note that $S'' \cup S' = S_d^c$), let

$$M = \max_{x\in S''}\{\frac{\partial V}{\partial z} f_0(z,\xi) + \xi q(z,\xi)\} \qquad m = \min_{x\in S''}\{b(z,\xi)\xi^2\}$$

and observe that $m > 0$ because $b(z,\xi) > 0$ and ξ cannot vanish at any point of S''.

Thus, since $k > 0$, we obtain

$$\dot{W}(x) \le M - km \qquad \forall x \in S''\,.$$

Let k_1 be such that $M - k_1 m = -a/2$. Then, if $k \ge k_1$,

$$\dot{W}(x) \le -a/2 \qquad \forall x \in S''\,. \tag{16}$$

This, together with (15) shows that

$$k \geq k_1 \qquad \Rightarrow \qquad \dot{W}(x) \leq -a/2 \qquad \forall x \in S_d^c\,,$$

as anticipated.

This being the case, suppose the initial condition $x(0)$ of (14) is in S_d^c. It follows from known arguments that $x(t) \in \Omega_c$ for all $t \geq 0$ and, at some time

$$T \leq 2(c-d)/a\,,$$

$x(T)$ is on the boundary of the set Ω_d.

On the boundary of Ω_d the derivative of $W(x(t))$ with respect to time is negative and hence the trajectory enters the set Ω_d and remains there for all $t \geq T$. Since all $x \in \Omega_d$ are such that $\|x\| < \epsilon$, this completes the proof.

The case of a system having relative degree $r > 1$ can be reduced to the case discussed above by means of a technique which is reminiscent of the technique of "adding stable zeros" used to similar purposes in linear systems.

Suppose the normal form

$$\begin{aligned} \dot{z} &= f_0(z, \xi_1, \ldots, \xi_{r-1}, \xi_r) \\ \dot{\xi}_1 &= \xi_2 \\ &\cdots \\ \dot{\xi}_{r-1} &= \xi_r \\ \dot{\xi}_r &= q(z, \xi_1, \ldots, \xi_{r-1}, \xi_r) + b(z, \xi_1, \ldots, \xi_{r-1}, \xi_r)u \end{aligned} \quad . \qquad (17)$$

is globally defined and let the variable ξ_r be replaced by a new state variable defined as

$$\theta = \xi_r + a_0\xi_1 + a_1\xi_2 + \cdots + a_{r-2}\xi_{r-1}$$

in which $a_0, a_1, \ldots, a_{r-2}$ are design parameters.

After this change of coordinates, the system becomes

$$
\begin{aligned}
\dot{z} &= f_0(z, \xi_1, \ldots, \xi_{r-1}, -\textstyle\sum_{i=1}^{r-1} a_{i-1}\xi_i + \theta) \\
\dot{\xi}_1 &= \xi_2 \\
&\cdots \\
\dot{\xi}_{r-2} &= \xi_{r-1} \\
\dot{\xi}_{r-1} &= -\textstyle\sum_{i=1}^{r-1} a_{i-1}\xi_i + \theta \\
\dot{\theta} &= a_0\xi_2 + a_1\xi_3 + \cdots + a_{r-2}(-\textstyle\sum_{i=1}^{r-1} a_{i-1}\xi_i + \theta) \\
&\quad + q(z, \xi_1, \ldots, \xi_{r-1}, -\textstyle\sum_{i=1}^{r-1} a_{i-1}\xi_i + \theta) \\
&\quad + b(z, \xi_1, \ldots, \xi_{r-1}, -\textstyle\sum_{i=1}^{r-1} a_{i-1}\xi_i + \theta)u\,.
\end{aligned}
$$

This system, with θ regarded as output, has a structure which is identical to that of system (11)

In fact, if we set

$$
\begin{aligned}
\zeta &= \mathrm{col}(z, \xi_1, \ldots, \xi_{r-1}) \in \mathbb{R}^{n-1} \\
\bar{y} &= \theta
\end{aligned}
$$

we obtain a system of the form

$$
\begin{aligned}
\dot{\zeta} &= \bar{f}_0(\zeta, \theta) \\
\dot{\theta} &= \bar{q}(\zeta, \theta) + \bar{b}(\zeta, \theta)u \\
\bar{y} &= \theta\,.
\end{aligned}
\tag{18}
$$

In order to be able to use the stabilization results deduced above, it remains to check whether system (18) is globally minimum-phase (respectively, globally – and also locally exponentially – minimum-phase) i.e. whether the equilibrium $\zeta = 0$ of

$$\dot{\zeta} = \bar{f}_0(\zeta, 0)$$

is globally asymptotically stable (respectively, globally asymptotically and locally exponentially stable).

This system has the structure of a cascade interconnection

$$\begin{aligned}
\dot{z} &= f_0(z, \xi_1, \ldots, \xi_{r-1}, -\textstyle\sum_{i=1}^{r-1} a_{i-1}\xi_i) \\
\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \cdots \\ \dot{\xi}_{r-2} \\ \dot{\xi}_{r-1} \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{r-2} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \cdots \\ \xi_{r-2} \\ \xi_{r-1} \end{pmatrix}.
\end{aligned} \tag{19}$$

If the a_i's are such that the polynomial

$$d(\lambda) = \lambda^{r-1} + a_{r-2}\lambda^{r-2} + \cdots + a_1\lambda + a_0 \tag{20}$$

is Hurwitz, the lower subsystem of the cascade is (globally) asymptotically stable. If system (17) is strongly minimum-phase, the upper subsystem of the cascade, viewed as a system with input $(\xi_1, \ldots, \xi_{r-1})$ and state z is input-to-state stable. Thus, system (19) is globally asymptotically stable.

If, in addition, the zero dynamics of (17) are also locally exponentially stable, then system (19) is also locally exponentially stable.

As far as the stabilizing law is concerned, observe that – in the present context – the stabilizing feedback (12) becomes

$$u = -k\theta = -k(a_0\xi_1 + a_1\xi_2 + \cdots + a_{r-2}\xi_{r-1} + \xi_r)$$

Bearing in mind the fact that $\xi_i = L_f^{i-1}h(x)$ for $i = 1, \ldots, r$, one can conclude that the stabilizing feedback as the form

$$u = -k\Big(\sum_{i=1}^{r} a_i L_f^{i-1} h(x)\Big) . \qquad (21)$$

with the a_i's such that the polynomial (20) is Hurwitz and $a_{r-1} = 1$.

Proposition For every choice of a compact set $\mathcal{C}$ and of a number $\varepsilon > 0$, there is a number k^* and a finite time T such that, if $k \geq k^*$, all trajectories of the closed-loop system with initial condition $x(0) \in \mathcal{C}$ remain bounded and satisfy $\|x(t)\| < \varepsilon$ for all $t \geq T$. If the system is strongly – and also locally exponentially – minimum-phase, for every choice of a compact set $\mathcal{C}$ there is a number k^* such that, if $k \geq k^*$, the equilibrium $x = 0$ of the resulting closed-loop system is asymptotically (and locally exponentially) stable, with a domain of attraction $\mathcal{A}$ that contains the set $\mathcal{C}$.