A Summary of Basic Concepts from the Theory of Stability

Comparison Functions

- A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$.

  If $a = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$, the function is said to belong to class $\mathcal{K}_\infty$.

  Examples

$$\alpha = Ar, \qquad \alpha = Ar^2, \qquad \alpha = A\sqrt{r}, \qquad \alpha(r) = \tan^{-1}(r)$$

- A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed t, the function

$$\begin{array}{rccc} \alpha : & [0, a) & \to & [0, \infty) \\ & r & \mapsto & \beta(r, t) \end{array}$$

  belongs to class $\mathcal{K}$ and, for each fixed r, the function

$$\begin{array}{rccc} \varphi : & [0, \infty) & \to & [0, \infty) \\ & t & \mapsto & \beta(r, t) \end{array}$$

  is decreasing and $\lim_{t\to\infty} \varphi(t) = 0$.

  Examples

$$\beta(r, t) = Are^{-\lambda t}, \qquad \beta(r, t) = Ar^2e^{-\lambda t}, \qquad \beta(r, t) = \frac{A\sqrt{r}}{1 + t^2}$$

Consider an autonomous nonlinear system

$$\dot{x} = f(x) \tag{1}$$

in which $x \in \mathbb{R}^n$, $f(0) = 0$ and $f(x)$ is locally Lipschitz. The stability, or asymptotic stability, properties of the equilibrium $x = 0$ of this system can be tested via the well known criterion of Lyapunov.
Let B_d denote the open ball of radius d in $\mathbb{R}^n$, i.e.

$$B_d = \{x \in \mathbb{R}^n : \|x\| < d\} .$$

Theorem Let $V : B_d \to \mathbb{R}$ be a C^1 function such that, for some class $\mathcal{K}$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, defined on $[0, d)$,

$$\underline{\alpha}(\|x\|) \le V(x) \le \overline{\alpha}(\|x\|) \qquad \text{for all } x \in B_d. \tag{2}$$

- If
$$\frac{\partial V}{\partial x} f(x) \le 0 \qquad \text{for all } x \in B_d, \tag{3}$$
the equilibrium $x = 0$ of (1) is stable.
- If, for some class $\mathcal{K}$ function $\alpha(\cdot)$, defined on $[0, d)$,
$$\frac{\partial V}{\partial x} f(x) \le -\alpha(\|x\|) \qquad \text{for all } x \in B_d, \tag{4}$$
the equilibrium $x = 0$ of (1) is locally asymptotically stable.
- If $d = \infty$ and, in the above inequalities, $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$ are class $\mathcal{K}_\infty$ functions, the equilibrium $x = 0$ of (1) is globally asymptotically stable.

Sometimes, in the design of feedback laws, while it is difficult to obtain a system whose equilibrium $x = 0$ is globally asymptotically stable, it is relatively more easy to obtain a system in which trajectories are bounded (maybe for a specific set of initial conditions) and have suitable decay properties.

Instrumental, in such context, is the notion of sublevel set of a Lyapunov function $V(x)$ which, for a fixed non-negative real number c, is defined as

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}.$$

As an example of how sublevel sets can be used to analyze boundedness and decay of trajectories, consider the following.

Let r_1 and r_2 be two positive numbers, with $r_2 > r_1$. Suppose $V(x)$ is a function satisfying (2), with $\underline{\alpha}(\cdot)$ a class $\mathcal{K}_\infty$ function. Pick any pair of positive numbers c_1, c_2, such that

$$\Omega_{c_1} \subset B_{r_1} \subset B_{r_2} \subset \Omega_{c_2} .$$

Let $S_{c_1}^{c_2}$ denote the "annular" compact set

$$S_{c_1}^{c_2} = \{x \in \mathbb{R}^n : c_1 \leq V(x) \leq c_2\} .$$

Suppose that, for some $a > 0$,

$$\frac{\partial V}{\partial x} f(x) \leq -a \qquad \text{for all } x \in S_{c_1}^{c_2} .$$

Then, for each initial condition $x(0) \in B_{r_2}$, the trajectory $x(t)$ of (1) is defined for all t and there exists a finite time T such that $x(t) \in B_{r_1}$ for all $t \geq T$.

In fact, take any $x(0) \in B_{r_2} \setminus \Omega_{c_1}$. Such $x(0)$ is in $S_{c_1}^{c_2}$. So long as $x(t) \in S_{c_1}^{c_2}$, the function $V(x(t))$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \leq -a$$

and hence

$$V(x(t)) \leq V(x(0)) - at \leq c_2 - at\,.$$

Thus, at a time $T \leq (c_2 - c_1)/a$, $x(T)$ is on the boundary of the set Ω_{c_1}.

On the boundary of Ω_{c_1} the derivative of $V(x(t))$ with respect to time is negative and hence the trajectory enters the set Ω_{c_1} and remains there for all $t \geq T$.

Input-to-state stability

In the analysis of forced nonlinear systems, the property of input-to-state stability, introduced and thoroughly studied by E.D. Sontag, plays a role of paramount importance.

Consider a forced nonlinear system

$$\dot{x} = f(x, u) \tag{5}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, in which $f(0,0) = 0$ and $f(x,u)$ is locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$.

The input function $u : [0, \infty) \to \mathbb{R}^m$ of (5) can be any piecewise continuous bounded function. The space of all such functions is endowed with the so-called supremum norm $\|u(\cdot)\|_{[0,t)}$, which is defined as

$$\|u(\cdot)\|_{[0,t)} = \sup_{\tau \in [0,t)} \|u(\tau)\| .$$

Definition System (5) is said to be input-to-state stable if there exist a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class $\mathcal{K}$ function $\gamma(\cdot)$, called a gain function, such that, for any bounded input $u(\cdot)$ and any $x(0) \in \mathbb{R}^n$, the response $x(t)$ of (5) in the initial state $x(0)$ satisfies

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\|u(\cdot)\|_{[0,t)}) \tag{6}$$

for all $t \geq 0$.

An alternative way to say that a system is input-to-state stable is to say that there exists a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that, for any bounded input $u(\cdot)$ and any $x(0) \in \mathbb{R}^n$, the response $x(t)$ of (5) in the initial state $x(0)$ satisfies

$$\|x(t)\| \le \max\{\beta(\|x(0)\|, t), \gamma(\|u(\cdot)\|_{[0,t)}\} \quad \text{for all } t \ge 0 \tag{7}$$

Note that input-to-state stability implies global asymptotic stability of the equilibrium $z = 0$ of $\dot{x} = f(x, 0)$.

A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is called an ISS-Lyapunov function for system (5) if there exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and a class $\mathcal{K}$ function $\chi(\cdot)$ such that

$$\underline{\alpha}(\|x\|) \le V(x) \le \overline{\alpha}(\|x\|) \quad \text{for all } x \in \mathbb{R}^n \tag{8}$$

and

$$\frac{\partial V}{\partial x} f(x,u) \le -\alpha(\|x\|) \quad \text{for all } (x,u) \in \mathbb{R}^n \times \mathbb{R}^m \text{ satisfying } \|x\| \ge \chi(\|u\|). \tag{9}$$

An alternative, equivalent, definition is the following one. A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is an ISS-Lyapunov function for system (5) if and only if there exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and a class $\mathcal{K}$ function $\sigma(\cdot)$ such that (8) holds and

$$\frac{\partial V}{\partial x} f(x,u) \le -\alpha(\|x\|) + \sigma(\|u\|) \qquad \text{for all } (x,u) \in \mathbb{R}^n \times \mathbb{R}^m\,. \tag{10}$$

Theorem System (5) is input-to-state stable if and only if there exists an ISS-Lyapunov function. In particular, if such function exists, then an estimate of the form (6) holds with $\gamma(r) = \underline{\alpha}^{-1}(\overline{\alpha}(\chi(r)))$.

Example: A stable linear system

$$\dot{x} = Ax + Bu$$

is input-to-state stable, with a linear gain function.

Example: The system

$$\dot{x} = -ax^k + bx^p u$$

with $a > 0$ and $k > p$ is input-to-state stable, with gain function

$$\gamma(r) = A\, r^{\frac{1}{k-p}}\,.$$

Counterexample: The system

$$\dot{x} = -x + xu$$

is **not** input-to-state stable.

In the design of stabilizing feedback laws, it often occurs to determine the stability of a cascade-connected system of the form

$$\begin{aligned} \dot{z} &= f(z,\xi) \\ \dot{\xi} &= g(\xi)\,, \end{aligned} \tag{11}$$

in which $z \in \mathbb{R}^n$, $\xi \in \mathbb{R}^m$, $f(0,0) = 0$, $g(0) = 0$.

Lemma Suppose the equilibrium $z = 0$ of

$$\dot{z} = f(z,0) \tag{12}$$

is locally asymptotically stable and the equilibrium $\xi = 0$ of $\dot{\xi} = g(\xi)$ is stable. Then the equilibrium $(z,\xi) = (0,0)$ of (11) is stable. If the equilibrium $\xi = 0$ of $\dot{\xi} = g(\xi)$ is locally asymptotically stable, then the equilibrium $(z,\xi) = (0,0)$ of (11) is locally asymptotically stable.

It should be stressed, however, that the global asymptotic stability of $z = 0$ as an equilibrium of (12) and the global asymptotic stability of $\xi = 0$ as an equilibrium of $\dot{\xi} = g(\xi)$ **do not** imply, in general, global asymptotic stability of the equilibrium $(z,\xi) = (0,0)$ of the cascade.

Example: Consider the system

$$\begin{aligned} f(z,\xi) &= -z + z^2\xi \\ g(\xi) &= -\xi\,. \end{aligned}$$

Clearly $z = 0$ is a globally asymptotically equilibrium of $\dot{z} = f(z,0)$ and $\xi = 0$ is a globally asymptotically equilibrium of $\dot{\xi} = g(\xi)$. However, this system has finite escape times.

Theorem Suppose that system

$$\dot{z} = f(z,\xi)\,, \tag{13}$$

viewed as a system with input ξ and state z is input-to-state stable and that system

$$\dot{\xi} = g(\xi,u)\,, \tag{14}$$

viewed as a system with input u and state ξ is input-to-state stable as well. Then, system

$$\begin{aligned} \dot{z} &= f(z,\xi) \\ \dot{\xi} &= g(\xi,u) \end{aligned}$$

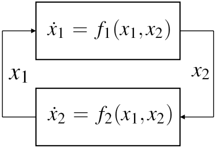
is input-to-state stable.

Feedback-connected systems

Consider a nonlinear system modeled by equations of the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2)\ , \end{aligned} \tag{15}$$

in which $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, and $f_1(0,0) = 0$, $f_2(0,0) = 0$ (see Fig. 11). This is seen as interconnection of a system Σ_1 with internal state x_1 and input x_2 and of a system Σ_2 with internal state x_2 and input x_1.



Assume that both Σ_1 and Σ_2 are input-to-state stable.

This means that there exists two class $\mathcal{KL}$ functions $\beta_1(\cdot,\cdot), \beta_2(\cdot,\cdot)$ and two class $\mathcal{K}$ functions $\gamma_1(\cdot), \gamma_2(\cdot)$ such that,

for any bounded input $x_2(\cdot)$ and any $x_1(0) \in \mathbb{R}^{n_1}$, the response $x_1(t)$ of

$$\dot{x}_1 = f_1(x_1, x_2)$$

in the initial state $x_1(0)$ satisfies

$$\|x_1(t)\| \le \max\{\beta_1(\|x_1(0)\|, t), \gamma_1(\|x_2(\cdot)\|_{[0,t]})\} \tag{16}$$

for all $t \ge 0$,

and for any bounded input $x_1(\cdot)$ and any $x_2(0) \in \mathbb{R}^{n_2}$, the response $x_2(t)$ of

$$\dot{x}_2 = f_2(x_1, x_2)$$

in the initial state $x_2(0)$ satisfies

$$\|x_2(t)\| \le \max\{\beta_2(\|x_2(0)\|, t), \gamma_2(\|x_1(\cdot)\|_{[0,t]})\} \tag{17}$$

for all $t \ge 0$.

It can be proven that ,if the composite function $\gamma_1 \circ \gamma_2(\cdot)$ satisfies [1]

$$\gamma_1 \circ \gamma_2(r) < r \quad \text{for all } r > 0\,, \tag{18}$$

the pure feedback interconnection of Σ_1 and Σ_2 is globally asymptotically stable.

This result is usually referred to as the small-gain theorem for input-to-state stable systems.

Theorem Suppose Σ_1 and Σ_2 are two input-to-state stable systems. If the condition (18) holds, system (15)is globally asymptotically stable.

[1] A function $\gamma : [0, \infty) \to [0, \infty)$ satisfying $\gamma(0) = 0$ and $\gamma(r) < r$ for all $r > 0$ is called a simple contraction. Observe that if $\gamma_1 \circ \gamma_2(\cdot)$ is a simple contraction, then also $\gamma_2 \circ \gamma_1(\cdot)$ is a simple contraction. In fact, let $\gamma_1^{-1}(\cdot)$ denote the inverse of the function $\gamma_1(\cdot)$, which is defined on an interval of the form $[0, r_1^*)$ where

$$r_1^* = \lim_{r\to\infty} \gamma_1(r)\ .$$

If $\gamma_1 \circ \gamma_2(\cdot)$ is a simple contraction, then

$$\gamma_2(r) < \gamma_1^{-1}(r) \ \text{ for all } \ 0 < r < r_1^*\ ,$$

and this shows that

$$\gamma_2(\gamma_1(r)) < r \ \text{ for all } \ r > 0\ ,$$

i.e. $\gamma_2 \circ \gamma_1(\cdot)$ is a simple contraction.