Observers for nonlinear systems
Part II: The high-gain observer

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The high-gain construction

Letting $\underline{z}_i$ denote the vector

$$\underline{z}_i = \mathrm{col}(z_1, \ldots, z_i)$$

the canonical form in question can be rewritten in more concise form as

$$\begin{array}{rcl} \dot{z}_1 & = & f_1(\underline{z}_1, z_2, u) \\ \dot{z}_2 & = & f_2(\underline{z}_2, z_3, u) \\ & & \cdots \\ \dot{z}_{n-1} & = & f_{n-1}(\underline{z}_{n-1}, z_n, u) \\ \dot{z}_n & = & f_n(\underline{z}_n, u) \\ y & = & h(z_1, u)\,. \end{array} \tag{1}$$

The construction described below reposes on the following two additional technical assumptions:

(i) each of the maps $f_i(\underline{z}_i, z_{i+1}, u)$, for $i = 1, \ldots, n-1$, is globally Lipschitz with respect to $\underline{z}_i$, uniformly in z_{i+1} and u, and the map $f_n(\underline{z}_n, u)$ is globally Lipschitz with respect to $\underline{z}_n$, uniformly in u.

(ii) there exist two real numbers α, β, with $0 < \alpha < \beta$, such that

$$\alpha \le \left|\frac{\partial h}{\partial z_1}\right| \le \beta\,, \quad \text{and} \quad \alpha \le \left|\frac{\partial f_i}{\partial z_{i+1}}\right| \le \beta\,, \quad \text{for all} \quad i = 1, \ldots, n-1$$

for all $z \in \mathbb{R}^n$, and all $u \in \mathbb{R}^m$.

Note that the properties (i) and (ii) can be assumed without loss of generality if it is known – a priori – that $z(t)$ remains in a compact set $\mathcal{C}$.

The observer for (1) consists of a copy of the dynamics of (1) corrected by an innovation term proportional to the difference between the output of (1) and the output of the copy.

More precisely, the observer in question is a system of the form

$$\begin{array}{rcl} \dot{\hat{z}}_1 & = & f_1(\hat{\underline{z}}_1, \hat{z}_2, u) + \kappa c_{n-1}(y - h(\hat{z}_1, u)) \\ \dot{\hat{z}}_2 & = & f_2(\hat{\underline{z}}_2, \hat{z}_3, u) + \kappa^2 c_{n-2}(y - h(\hat{z}_1, u)) \\ & & \cdots \\ \dot{\hat{z}}_{n-1} & = & f_{n-1}(\hat{\underline{z}}_{n-1}, \hat{z}_n, u) + \kappa^{n-1} c_1(y - h(\hat{z}_1, u)) \\ \dot{\hat{z}}_n & = & f_n(\hat{\underline{z}}_n, u) + \kappa^n c_0(y - h(\hat{z}_1, u)) \, , \end{array} \tag{2}$$

in which κ and $c_{n-1}, c_{n-2}, \ldots, c_0$ are design parameters.

Define a rescaled observation error defined as

$$e_i == \kappa^{n-i}(z_i - \hat{z}_i), \qquad i = 1, 2, \ldots, n\,. \tag{3}$$

Appropriate calculation show that

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \cdots \\ \dot{e}_{n-1} \\ \dot{e}_n \end{pmatrix} = \kappa \begin{pmatrix} -c_{n-1}g_1(t) & g_2(t) & 0 & \cdots & 0 & 0 \\ -c_{n-2}g_1(t) & 0 & g_3(t) & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ -c_1 g_1(t) & 0 & 0 & \cdots & 0 & g_n(t) \\ -c_0 g_1(t) & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \cdots \\ e_{n-1} \\ e_n \end{pmatrix} + \begin{pmatrix} \kappa^{n-1}F_1 \\ \kappa^{n-2}F_2 \\ \cdots \\ \kappa F_{n-1} \\ F_n \end{pmatrix}. \tag{4}$$

in which

$$F_i = f_i(\underline{z}_i(t), \hat{z}_{i+1}(t), u(t)) - f_i(\underline{\hat{z}}_i(t), \hat{z}_{i+1}(t), u(t))$$

for $i = 1, 2, \ldots, n-1$ and

$$F_n = f_n(\underline{z}_n(t), u(t)) - f_n(\underline{\hat{z}}_n(t), u(t))\,.$$

The terms $g_i(t)$'s are defined as

$$\begin{aligned} g_1(t) &= \frac{\partial h}{\partial z_1}(\delta_0(t), u(t)) \\ g_{i+1}(t) &= \frac{\partial f_i}{\partial z_{i+1}}(\underline{z}_i(t), \delta_i(t), u(t)) \qquad \text{for } i = 1, \ldots, n-1, \end{aligned} \tag{5}$$

in which the $\delta_i(t)$'s satisfy

$$\delta_i(t) \in [\hat{z}_{i+1}(t), z_{i+1}(t)] \,.$$

Note that

$$\alpha \le |g_i(t)| \le \beta$$

The error equation can be written in compact form as

$$\dot{e} = A(t)e + \tilde{F}(t)$$

in which

$$A(t) = \begin{pmatrix} -c_{n-1}g_1(t) & g_2(t) & 0 & \cdots & 0 & 0 \\ -c_{n-2}g_1(t) & 0 & g_3(t) & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ -c_1 g_1(t) & 0 & 0 & \cdots & 0 & g_n(t) \\ -c_0 g_1(t) & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and

$$\tilde{F}(t) = \mathrm{col}(\kappa^{n-1}F_1, \kappa^{n-2}F_2, \ldots, F_n)\,.$$

It can be shown that, for some L,

$$\|\tilde{F}(t)\| \le L\sqrt{n}\|e\|\,.$$

Lemma Consider a matrix of the form $A(t)$ indicated above and suppose there exists two real numbers α, β, with $0 < \alpha < \beta$, such that

$$\alpha \leq g_i(t) \leq \beta \qquad \text{for all } t \geq 0 \text{ and } i = 1, 2, \ldots, n\,. \tag{6}$$

Then, there is a set of real numbers $c_0, c_1, \ldots, c_{n-1}$, a real number $\lambda > 0$ and a symmetric positive definite $n \times n$ matrix S, all depending only on α and β, such that

$$SA(t) + A^T(t)S \leq -\lambda I\,. \tag{7}$$

As a consequence, for the Lyapunov function $V(e) = e^T Se$ we have

$$\begin{array}{rcl} \dot{V}(e(t)) & = & \kappa e^T(t)[A(t)S + SA^T(t)]e(t) + 2e^T(t)S\tilde{F}(t) \\ & \leq & -(\kappa\lambda - 2\|S\|L\sqrt{n})\|e\|^2 \end{array}$$

If κ is large enough, we have

$$\dot{V}(e(t)) \leq -aV(e(t))$$

for some $a > 0$ and hence

$$\lim_{t\to\infty} e(t) = 0\,.$$

Theorem Consider a nonlinear system in uniform observability canonical form (1) and an observer defined as in (2). Suppose assumptions (i) and (ii) hold. Then, there is a choice of the coefficients $c_0, c_1, \ldots, c_{n-1}$ and a value κ^* such that, if $\kappa > \kappa^*$,

$$\lim_{t\to\infty} (z_i(t) - \hat{z}_i(t)) = 0 \qquad \text{for all } i = 1, 2, \ldots, n$$

for any pair of initial conditions $(z(0), \hat{z}(0)) \in \mathbb{R}^n \times \mathbb{R}^n$.