Stabilization by Output Feedback
Part I: A nonlinear separation principle

Consider a single-input single-output nonlinear system in observability canonical form, which we rewrite in compact form as

$$\begin{array}{rcl} \dot{z} & = & f(z,u) \\ y & = & h(z,u)\,, \end{array} \tag{1}$$

with $f(0,0) = 0$ and $h(0,0) = 0$

Suppose there exists a feedback law $u = u^*(z)$, with $u^*(0) = 0$, such that the equilibrium $z = 0$ of

$$\dot{z} = f(z, u^*(z)) \tag{2}$$

is globally asymptotically stable.

Assume, for the time being, that the Assumptions (i) and (ii) hold (we shall see later how these can be removed) and consider an high-gain observer, which we rewrite in compact form as

$$\dot{\hat{z}} = f(\hat{z}, u) + D_\kappa G_0(y - h(\hat{z}, u))\,, \tag{3}$$

in which D_κ is the matrix

$$D_\kappa = \mathrm{diag}(\kappa, \kappa^2, \ldots, \kappa^n) \tag{4}$$

and G_0 is the vector

$$G_0 = \mathrm{col}(c_{n-1}, \ldots, c_1, c_0)\,.$$

An obvious choice to achieve asymptotic stability, suggested by the analogy with linear systems, would be to replace z by its estimate $\hat{z}$ in the map $u^*(z)$.

However, this simple choice may prove to be dangerous, for the following reason.

Bearing in mind the analysis carried out before, recall that

$$e = \begin{pmatrix} \kappa^{n-1} & \cdots & 0 & 0 \\ \cdot & \cdots & \cdot & \cdot \\ 0 & \cdots & \kappa & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} (z - \hat{z}) = \kappa^n D_\kappa^{-1}(z - \hat{z})\,.$$

We have seen that, to secure asymptotic convergence of the observation error $e(t)$ to zero is necessary to increase κ.

This, even if the initial conditions $z(0)$ and $\hat{z}(0)$ of the plant and of the observer are taken in a compact set, may entail large values of $e(t)$ for (positive) times close to $t = 0$.

In fact,

$$e(0) = \kappa^n D_\kappa^{-1}(z(0) - \hat{z}(0))$$

and $\|e(0)\|$ grows unbounded with increasing κ.

Since $e(t)$ is a continuous function of t, we should expect that, if κ is large, there is an initial interval of time on which $\|e(t)\|$ is large. This phenomenon is sometimes referred to as "peaking".

Now, note that feeding the system (1) with a control $u = u^*(\hat{z})$ would result in a system

$$\dot{z} = f(z, u^*(z - \kappa^{-n} D_\kappa e)) .$$

This is viewed as a system with state z subject to an input $\kappa^{-n} D_\kappa e(t)$.

Now, if κ is large, the matrix $\kappa^{-n} D_\kappa$ remains bounded, because all elements of this (diagonal) matrix are non positive powers of κ. In fact $\|\kappa^{-n} D_\kappa\| = 1$ if $\kappa \geq 1$, as an easy calculation shows.

However, has remarked above, $\|e(t)\|$ may become large for small values of t, if κ is large. Since the system is nonlinear, this may result in a finite escape time.

To avoid such inconvenience, as a precautionary measure, it is appropriate to "saturate" the control, by choosing instead a law of the form

$$u = g_\ell(u^*(\hat{z}))) \tag{5}$$

in which $g_\ell : \mathbb{R} \to \mathbb{R}$ is a smooth *saturation* function, that is a function characterized by the following properties:

(i) $g_\ell(s) = s$ if $|s| \leq \ell$,
(ii) $g_\ell(s)$ is odd and monotonically increasing, with $0 < g'_\ell(s) \leq 1$,
(iii) $\lim_{s\to\infty} g_\ell(s) = \ell(1 + c)$ with $0 < c \ll 1$.

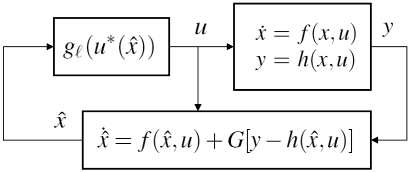
The real number $\ell > 0$ is usually referred to as the *saturation level*.

The consequence of choosing the control u as in (5) is that *global* asymptotic stability is no longer assured. However, as it will be shown, *semiglobal stabilizability* is still possible.

In fact, it can be showns, for every compact set $\mathcal{C}$ of initial conditions in the state space, there is a choice of design parameters such that the equilibrium $(z, \hat{z}) = (0, 0)$ of the closed loop system is asymptotically stable, with a domain of attraction that contains $\mathcal{C}$.

The proposed control structure has the following form

$$\begin{aligned} \dot{z} &= f(z, g_\ell(u^*(\hat{z}))) \\ \dot{\hat{z}} &= f(\hat{z}, g_\ell(u^*(\hat{z}))) + D_\kappa G_0[h(z, g_\ell(u^*(\hat{z}))) - h(\hat{z}, g_\ell(u^*(\hat{z})))] \,. \end{aligned} \tag{6}$$



Replacing $\hat{z}$ by its expression in terms of z and e, we obtain for the first equation a system that can be written as

$$\begin{array}{rcl} \dot{z} & = & f(z, g_\ell(u^*(z - \kappa^{-n} D_\kappa e))) \\ & = & f(z, u^*(z)) - f(z, u^*(z)) + f(z, g_\ell(u^*(z - \kappa^{-n} D_\kappa e))) \\ & = & F(z) + \Delta(z, e) \end{array}$$

in which

$$\begin{array}{rcl} F(z) & = & f(z, u^*(z)) \\ \Delta(z, e) & = & f(z, g_\ell(u^*(z - \kappa^{-n} D_\kappa e))) - f(z, u^*(z))\,. \end{array}$$

Note that, by the inverse Lyapunov theorem, since the equilibrium $z = 0$ of (2) is globally asymptotically stable, there exists a smooth function $V(z)$, satisfying

$$\underline{\alpha}(\|z\|) \le V(z) \le \overline{\alpha}(\|z\|) \qquad \text{and} \qquad \frac{\partial V}{\partial z} F(z) \le -\alpha(\|z\|) \qquad \text{for all } z,$$

for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot), \overline{\alpha}(\cdot), \alpha(\cdot)$.

For simplicity, let the compact set $\mathcal{C}$ in which initial conditions are taken be a set of the form $\mathcal{C} = \overline{B}_R \times \overline{B}_R$ in which $\overline{B}_R$ denotes the closure of B_R, the ball of radius R in $\mathbb{R}^n$.

Choose a number c such that

$$\Omega_c = \{z \in \mathbb{R}^n : V(z) \le c\} \supset \overline{B}_R\,,$$

and then choose the parameter ℓ in the definition of $g_\ell(\cdot)$ as

$$\ell = \max_{z \in \Omega_{c+1}} u^*(z) + 1\,.$$

Since e enters in $\Delta(z, e)$ through the bounded function $g_\ell(\cdot)$, it is easy to realize that there is a number δ_1 such that

$$\|\Delta(z, e)\| \le \delta_1, \qquad \text{for all } z \in \Omega_{c+1} \text{ and all } e \in \mathbb{R}^n\,.$$

Note also that, if $z \in \Omega_{c+1}$ and $\|\kappa^{-n} D_\kappa e\|$ is small,

$$g_\ell(u^*(z - \kappa^{-n} D_\kappa e)) = u^*(z - \kappa^{-n} D_\kappa e) \tag{7}$$

and hence $\Delta(z, 0) = 0$.

Assuming, without loss of generality, $\kappa \ge 1$, it is seen that $\|\kappa^{-n} D_\kappa\| = 1$ and hence $\|e\|$ small implies $\|\kappa^{-n} D_\kappa e\|$ small. Therefore, there are numbers δ_2, ε such that

$$\|\Delta(z, e)\| \le \delta_2 \|e\|, \qquad \text{for all } z \in \Omega_{c+1} \text{ and all } \|e\| \le \varepsilon\,.$$

These numbers $\delta_1, \delta_2, \varepsilon$ are independent of κ (so long as $\kappa \ge 1$) and only depend on the number R that characterizes the radius of the ball $\overline{B}_R$ in which $z(0)$ is taken.

Let $z(0) \in \overline{B}_R \subset \Omega_c$. Regardless of what $e(t)$ is, so long as $z(t) \in \Omega_{c+1}$, we have

$$\dot{V}(z(t)) = \frac{\partial V}{\partial z}[F(z) + \Delta(z, e)] \leq -\alpha(\|z\|) + \|\frac{\partial V}{\partial z}\|\delta_1 \,.$$

Setting

$$M = \max_{z \in \Omega_{c+1}} \|\frac{\partial V}{\partial z}\|$$

we observe that the previous estimate yields, in particular

$$\dot{V}(z(t)) \leq M\delta_1$$

which in turn yields

$$V(z(t)) \leq V(z(0)) + M\delta_1 t \leq c + M\delta_1 t \,.$$

From this it is deduced that $z(t)$ remains in Ω_{c+1} at least until time $T_0 = 1/M\delta_1$. This time may be very small but, because of the presence of the saturation function $g_\ell(\cdot)$, it is independent of κ. It rather only depends on the number R that characterizes the radius of the ball $\overline{B}_R$ in which $z(0)$ is taken.

Recall now that the variable e decays exponentially. Letting $V(e)$ denote the quadratic form $V(e) = e^T S e$, we know that

$$\dot{V}(e(t)) \leq -2\alpha_\kappa V(e(t))$$

in which

$$\alpha_\kappa = \frac{1}{2}(\kappa\lambda - 2\|S\|L\sqrt{n}) := \frac{1}{2}(\kappa\lambda - a_0)$$

is a number that can be made arbitrarily large by increasing κ (recall that λ and $\|S\|$ only depend of the bounds α and β in Assumption (ii) and L on the Lipschitz constants in Assumption (i)).

From this inequality, bearing in mind the fact that

$$a_1\|e\|^2 \leq V(e) \leq a_2\|e\|^2$$

in which a_1, a_2 are numbers depending on S and hence only on α, β, we obtain

$$\|e(t)\| \leq A\, e^{-\alpha_\kappa t}\, \|e(0)\|\,, \qquad \text{with } A = \sqrt{\frac{a_2}{a_1}}\,,$$

which is valid for all t, so long as $z(t)$ exists.

Recall now that $e(0) = \kappa^n D_\kappa^{-1}(z(0) - \hat{z}(0))$. Since $\kappa \geq 1$, it is seen that $\|\kappa^n D_\kappa^{-1}\| = \kappa^{n-1}$. Thus, if initial conditions are such that $(z(0), \hat{z}(0)) \in \overline{B}_R \times \overline{B}_R$, we have

$$\|e(0)\| \leq 2\kappa^{n-1} R\,.$$

Consequently

$$\|e(t)\| \leq 2AR\; e^{-\alpha_\kappa t} \kappa^{n-1}\,,$$

which is valid so long as $z(t)$ is defined. Note that

$$2AR\; e^{-\alpha_\kappa T_0} \kappa^{n-1} = 2AR\; e^{\frac{a_0 T_0}{2}} e^{-\frac{\lambda T_0}{2}\kappa} \kappa^{n-1}\,.$$

The function $e^{-\frac{\lambda T_0}{2}\kappa} \kappa^{n-1}$ is a polynomial function of κ multiplied by an exponentially decaying function of κ (recall that $T_0 > 0$). Thus, it tends to 0 as $\kappa \to \infty$. As a consequence, for any ε there is a number κ^* such that, if $\kappa > \kappa^*$,

$$\|e(T_0)\| \leq \varepsilon\,.$$

This also implies (since $\alpha_\kappa > 0$)

$$\|e(t)\| \leq 2AR\; e^{-\alpha_\kappa (t - T_0)} e^{-\alpha_\kappa T_0} \kappa^{n-1} \leq \varepsilon\,,$$

for all $t > T_0$, so long as $z(t)$ is defined.

Return now to the inequality

$$\dot{V}(z(t)) = \frac{\partial V}{\partial z}[F(z) + \Delta(z,e)] \leq -\alpha(\|z\|) + \|\frac{\partial V}{\partial z}\|\|\Delta(z,e)\|\,.$$

Pick $\kappa \geq \kappa^*$, so that $\| e(t)\| \leq \varepsilon$ for all $t \geq T_0$ and hence, so long as $z(t) \in \Omega_{c+1}$,

$$\dot{V}(z(t)) \leq -\alpha(\|z(t)\|) + M\delta_2\varepsilon\,.$$

Pick any number $d \ll c$ and consider the "annular" compact set

$$S_d^{c+1} = \{z : d \leq V(z) \leq c+1\}\,.$$

Let r be

$$r = \min_{z\in S_d^{c+1}} \|z\|\,.$$

By construction

$$\alpha(\|z\|) \geq \alpha(r) \qquad \text{for all } z \in S_d^{c+1}.$$

If ε is small enough

$$M\delta_2\varepsilon \leq \frac{1}{2}\alpha(r)\,,$$

and hence

$$\dot{V}(z(t)) \leq -\frac{1}{2}\alpha(r)\,,$$

so long as $z(t) \in S_d^{c+1}$.

By standard arguments, this proves that any trajectory $z(t)$ which starts in $\overline{B}_R$, in a finite time (which only depends on the choice of R and d), enters the set Ω_d and remains in this set thereafter.

Observing that for any (small) ϵ' there is a number d such that $\Omega_d \subset B_{\epsilon'}$, it can be concluded that, *for any choice of* $\epsilon' \ll R$ there exist a number κ^* and a time T^* such that, if $\kappa > \kappa^*$, all trajectories with initial conditions $(z(0), \hat{z}(0)) \in \overline{B}_R \times \overline{B}_R$ are bounded and satisfy

$$(z(t), \hat{z}(t)) \in B_{\epsilon'} \times B_{\epsilon'} \qquad \text{for all } t \geq T^* \,.$$

Moreover, $\lim_{t\to\infty} e(t) = 0$. If ϵ' is small enough, on the set $B_{\epsilon'} \times B_{\epsilon'}$, the first equation of (6) becomes (see (7))

$$\dot{z} = F(z) + f(z, u^*(z - \kappa^{-n} D_\kappa e)) - f(z, u^*(z) \,.$$

Thus, it is concluded that also $\lim_{t\to\infty} z(t) = 0$.

Proposition Consider system (1), assumed to be expressed in uniform observability canonical form, and suppose Assumptions (i) and (ii) hold. Suppose that a state feedback law $u = u^*(z)$ globally asymptotically stabilizes the equilibrium $z = 0$ of (2). Let the system be controlled by (5), in which $\hat{z}$ is provided by the observer (3). Then, for every choice of R, there exist a number ℓ and a number κ^* such that, if $\kappa > \kappa^*$, all trajectories of the closed-loop system with initial conditions in $\overline{B}_R \times \overline{B}_R$ are bounded and $\lim_{t\to\infty}(z(t), \hat{z}(t)) = (0, 0)$.

The nonlinear separation principle (proof)

It remains to discuss the role of the Assumptions (i) and (ii). Having proven that the trajectories of the system starting in $B_R \times B_R$ remain in a bounded region, it suffices to look for numbers α and β and a Lipschitz constant L making Assumptions (i) and (ii) valid *only on this bounded region*, which is always possible.