Data driven control

linear and nonlinear systems

Lecture 1

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Models and control

> *If physics is the science of understanding the physical environment, then* **control theory** *may be viewed as* **the science of modifying that environment** *[...] Control theory does not deal directly with physical reality but with* **mathematical models**.

Rudolf Kalman, Control Theory, *Encyclopædia Britannica*

$$\begin{aligned} x^+/\dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \qquad \begin{aligned} x^+/\dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}$$

Data-driven control

To offset the lack of "known" models by the use of data

Using data through the lenses of control theory

Control when the dynamics is "unknown"

If the model is unknown, there are a few approaches

System identification from data + control of the identified system

- G. Pillonetto et al. "Kernel methods in system identification, machine learning and function estimation: A survey". Automatica, 50(3):657–682, 2014.

Direct data-based control design

- M.C. Campi, A. Lecchini, and S.M. Savaresi. "Virtual reference feedback tuning: a direct method for the design of feedback controllers". Automatica, 38(8):1337-1346, 2002.

These lectures "Direct" design of controllers from data for "unknown" systems

Direct because the method returns controllers via data-dependent SDPs

The system is "unknown" but some priors are available

Data are collected to infer information about the dynamics

The method:
- works with perturbed data of low complexity
- provides analytical guarantees of correctness
- is based on basic tools of automatic control

Control when the dynamics is unknown

These lectures “Direct” design of controllers from data for “unknown” systems

Lec 1	Linear systems Unperturbed data of low complexity
Lec 2	Perturbed data
Lec 3	Extensions
Lec 4	Nonlinear systems

The lectures will present a personal perspective and will focus on a few selected papers (listed at the end of the lectures). A broader overview and a discussion of related work are discussed in those papers.

Outline Lecture 1

We will study 3 (data-driven) control problems

- Full measurements Stabilization of linear systems via static state feedback
- Optimality Linear Quadratic Regulation
- Partial measurements Stabilization of linear systems via dynamic output feedback

To introduce the main ideas, in Lecture 1 we consider the ideal case of unperturbed (noise-free) data and linear systems.

Before diving into the control design, we introduce the dataset and a concept that is at the core of these lectures.

Dynamical control systems

We focus our attention on systems of the form

$$x^+ = Ax + Bu$$

- $x \in \mathbb{R}^n$ (state) and $u \in \mathbb{R}^m$ (control)
- $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are unknown matrices

Focus on discrete-time systems (but we will also briefly remark on continuous-time systems later on)

Dataset

Information about the system's dynamics is obtained from a T-long dataset of input/state samples collected during (multiple) experiment(s)

$$\mathbb{D} := \{u(k), x(k)\}_{k=0}^{T}$$

where the samples satisfy

$$x(k+1) = Ax(k) + Bu(k), \quad k = 0, \ldots, T-1$$

Persistence of excitation

The approach to the design of controllers from data is inspired by nonparametric data-dependent representations of the unknown dynamics. To recall the origin of such a representation, we recall a notion of persistently exciting signals, which is useful to generate "rich" data.

Definition The sequence of input values $u : [0, T-1] \cap \mathbb{Z} \to \mathbb{R}^m$

$$u(0), u(1), \ldots, u(T-1)$$

is persistently exciting (PE) of order L if the Hankel matrix associated to it

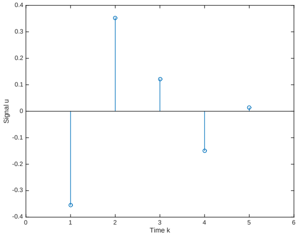
$$U_0 = \begin{bmatrix} u(0) & u(1) & \ldots & u(T-L) \\ u(1) & u(2) & \ldots & u(T-L+1) \\ \vdots & \vdots & \ddots & \vdots \\ u(L-1) & u(L) & \ldots & u(T-1) \end{bmatrix}$$

has full rank mL.

PE requires sufficiently long input sequences: $T \geq (m+1)L - 1$

Generating PE signals

```
global L m T ud

% Initializing the length of the probing input sequence

T=L*(m+1)-1;

% Generating the probing input sequence ud on [0,T-1]
% taking values in the interval [-0.5,0.5] in the form
% of an m x T matrix [ud(0) ud(1) .... ud(T-1)]

aux=zeros(m,T);
aux(:)=0.5;
ud(1:m,1:T)=rand(m,T)-aux;

% Computing the Hankel matrix Ud on [0,T-1]

for j=1:T-L+1
    for i=1:L
        Ud((i-1)*m+1:(i-1)*m+m,j)=ud(1:m, j+i-1);
    end
end

% If rank(Ud)= m*L then the sequence ud(0),...ud(T-1) is PE
% of order L as desired

if rank(Ud)== m*L
disp('input sequence is PE');
end
```



$L = 3, n = 2, m = 1 \Rightarrow T = 5$

$u_{[0,T-1]} =$
$[-0.355\ 0.353\ 0.1221\ -0.149\ 0.0132]$

$$U_0 = \begin{bmatrix} -0.3550 & 0.3530 & 0.1221 \\ 0.3530 & 0.1221 & -0.1490 \\ 0.1221 & -0.1490 & 0.0132 \end{bmatrix}$$

The Fundamental Lemma

A PE input applied to a linear reachable* system produces data that are sufficiently rich.

*A system is reachable if and only if $\operatorname{rank}\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n$

Lemma Let system

$$x(k+1) = Ax(k) + Bu(k)$$

be reachable. For any $t \geq 1$,

$$u_{[0,T-1]} \text{ PE of order } n+t \quad \Rightarrow \quad \operatorname{rank}\begin{bmatrix} U_0 \\ X_0 \end{bmatrix} = n + tm$$

where the matrix U_0 consists of the samples of the input sequence $u_{[0,T-1]} = \{u(0), u(1), \dots, u(T-1)\}$

$$U_0 = \begin{bmatrix} u(0) & u(1) & \dots & u(T-t) \\ u(1) & u(2) & \dots & u(T-t+1) \\ \vdots & \vdots & \ddots & \vdots \\ u(t-1) & u(t) & \dots & u(T-1) \end{bmatrix}$$

and the matrix X_0 consists of the samples of the state response $x(k+1) = Ax(k) + Bu(k)$, $k = 0, 1, \dots, T-t$, to the input sequence $u_{[0,T-1]}$

$$X_0 = \begin{bmatrix} x(0) & x(1) & \dots & x(T-t) \end{bmatrix}$$

J.C. Willems, P. Rapisarda, I. Markovsky, B.L. De Moor. "A note on persistency of excitation." Systems & Control Letters, 54, 4, 325–329, 2005.

Example

A partially known model ($n = 2$, $m = 1$, controllable system)

$u_{[0,T-1]}$ PE of order $n + t = 3$ ($n = 2$, $t = 1$), with $T = (n+t)(m+1) - 1 = 5$

$$u_{[0,T-1]} = \begin{bmatrix} -0.3550 & 0.3530 & 0.1221 & -0.1490 & 0.0132 \end{bmatrix}$$

We "experimentally" determine the matrix ($U_0 \in \mathbb{R}^{m \times T-t}, X_0 \in \mathbb{R}^{n \times T-t}$)

$$\begin{bmatrix} U_0 \\ \hline X_0 \end{bmatrix} = \left[\begin{array}{ccccc} -0.3550 & 0.3530 & 0.1221 & -0.1490 & 0.0132 \\ \hline 0.4027 & 0.3478 & 0.3571 & 0.3216 & 0.2362 \\ 0.4448 & 1.1451 & 1.7499 & 2.3708 & 2.9301 \end{array}\right]$$

where

$$X_0 = \begin{bmatrix} x(0) & x(1) & x(2) & x(3) & x(4) \end{bmatrix}$$

contains the state response of the system from the initial condition $x(0)$ to the input $u_{[0,4]}$. As predicted, $\begin{bmatrix} U_0 \\ X_0 \end{bmatrix}$ $\underline{\text{has rank } n + tm = 3}$.

Profound implications for control

$$\begin{bmatrix} U_0 \\ \hline X_0 \end{bmatrix} = \begin{bmatrix} u(0) & \ldots & u(T-t) \\ \vdots & \ddots & \vdots \\ u(t-1) & \ldots & u(T-1) \\ \hline x(0) & \ldots & x(T-t) \\ \vdots & \ddots & \vdots \\ x(t-1) & \ldots & x(T-1) \end{bmatrix}$$

$$\begin{bmatrix} \bar{u}(0) \\ \vdots \\ \bar{u}(t-1) \\ \hline \bar{x}(0) \\ \vdots \\ \bar{x}(t-1) \end{bmatrix} = \begin{bmatrix} U_0 \\ \hline X_0 \end{bmatrix} g$$

For a controllable linear system
(i) Let $u(0), \ldots, u(T-1)$ be PE of order $n+t$, $t \geq 1$, then any t-long input/state trajectory of the system $(\bar{u}_{[0,t-1]}, \bar{x}_{[0,t-1]})$ can be expressed as

$$\begin{bmatrix} \bar{u}_{[0,t-1]} \\ \bar{x}_{[0,t-1]} \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} g$$

where $g \in \mathbb{R}^{T-t+1}$.
(ii) Any linear combination of the columns of the matrix of data, i.e.,

$$\begin{bmatrix} U_0 \\ \hline X_0 \end{bmatrix} g,$$

is a t-long input-state trajectory of the system.

J. Coulson, J. Lygeros, F. Dörfler. "Data-Enabled Predictive Control: In the Shallows of the DeePC." *European Control Conference*, 2019.

Relating closed-loop trajectories with data

Consider Item (i) in the special case $t = 1$. Then

$$\begin{bmatrix} \bar{u}_{[0,t-1]} \\ \bar{x}_{[0,t-1]} \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} g \quad \text{becomes} \quad \begin{bmatrix} \bar{u}(0) \\ \bar{x}(0) \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} g$$

Given a $K \in \mathbb{R}^{m \times n}$, consider n 1-long input/state trajectories

$$\begin{bmatrix} \bar{u}(0) \\ \bar{x}(0) \end{bmatrix} = \begin{bmatrix} K\bar{x}(0) \\ \bar{x}(0) \end{bmatrix}, \quad \bar{x}(0) = e_i, \quad i = 1, 2, \ldots, n$$

where e_i is the i-th vector of the canonical basis of $\mathbb{R}^n$.

Then

$$\begin{bmatrix} K \\ I_n \end{bmatrix} \begin{bmatrix} e_1 & \ldots & e_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} \begin{bmatrix} g_1 & \ldots & g_n \end{bmatrix}$$

that is,

$$\boxed{\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G}$$

Stabilization of linear systems

Data-dependent representations

Consider the dataset

$$\mathbb{D} = \{u(k), x(k)\}_{k=0}^{T}, \quad x(k+1) = Ax(k) + Bu(k), \quad k = 0, \ldots, T-1$$

and store it into matrices U_0, X_0, X_1 defined as

$$\begin{aligned} U_0 &:= \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix} \\ X_0 &:= \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix} \\ X_1 &:= \begin{bmatrix} x(1) & x(2) & \cdots & x(T) \end{bmatrix} \end{aligned}$$

which satisfy the identity

$$\begin{aligned} &\underbrace{\begin{bmatrix} x(1) & x(2) & \ldots & x(T) \end{bmatrix}}_{X_1} \\ = \quad & A \underbrace{\begin{bmatrix} x(0) & x(1) & \ldots & x(T-1) \end{bmatrix}}_{X_0} + B \underbrace{\begin{bmatrix} u(0) & u(1) & \ldots & u(T-1) \end{bmatrix}}_{U_0} \end{aligned}$$

$$\boxed{X_1 = AX_0 + BU_0}$$

Data-dependent representations

Consider a full-state feedback $u = Kx$ and the resulting closed-loop system $x^+ = (A+BK)x$

Consider any matrices $K \in \mathbb{R}^{m\times n}$, $G_K \in \mathbb{R}^{T\times n}$ such that[1]

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G_K$$

where

$$\begin{aligned} U_0 &= \begin{bmatrix} u(0) & u(1) & \dots & u(T-1) \end{bmatrix} \\ X_0 &= \begin{bmatrix} x(0) & x(1) & \dots & x(T-1) \end{bmatrix} \end{aligned} \qquad X_1 = AX_0 + BU_0$$

The matrix $A+BK$ of the closed-loop system $x^+ = (A+BK)x$ is arranged as

$$\begin{aligned} & A+BK \\ = \; & \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K \\ I_n \end{bmatrix} \\ \overset{\left[\begin{smallmatrix} K \\ I_n \end{smallmatrix}\right] = \left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right] G_K}{=} \; & \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G_K \\ \overset{X_1 = \left[\begin{smallmatrix} B & A \end{smallmatrix}\right] \left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right]}{=} \; & X_1 G_K \end{aligned}$$

[1] If $\left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right]$ has full row rank, then for any K a G_K exists by Rouché-Capelli theorem.

Data-based parametrization of the closed-loop system

Consider system

$$x^+ = Ax + Bu$$

in closed-loop with a state feedback $u = Kx$. Consider any matrices $K \in \mathbb{R}^{m\times n}$, $G_K \in \mathbb{R}^{T\times n}$ such that

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G_K$$

Then the closed-loop system $x^+ = (A + BK)x$ has the following equivalent representation

$$x^+ = X_1 G_K x$$

- ▷ The representation depends on data U_0, X_0, X_1 and design variables G_K
- ▷ The design of the controller is shifted from K to G_K and in the process the system's matrices are replaced by data.
- ▷ If the system is controllable and the input PE of order $n+1$, rank $\begin{bmatrix} U_0 \\ X_0 \end{bmatrix} = n + m$ and matrices $K \in \mathbb{R}^{m\times n}$, $G_K \in \mathbb{R}^{T\times n}$ such that $\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G_K$ exist.

C. De Persis, P. Tesi. "Formulas for data-driven control: stabilization, optimality, robustness". IEEE Transactions on Automatic Control, 65, 3, 909–924, 2020.

Data-based stabilization

LMIs

A **linear matrix inequality** (LMI) is an expression of the form

$$F(y) := F_0 + F_1 y_1 + \ldots + F_N y_N \prec 0$$

where

- $F\colon \mathbb{R}^N \to \mathbb{S}^{M\times M}$ is an affine function
- $y = \begin{bmatrix} y_1 \ldots y_N \end{bmatrix}^\top \in \mathbb{R}^N$ is the variable
- $F_0, F_1, \ldots, F_N$ are **symmetric** matrices
- $F(y) \prec 0$ means that $F(y)$ is **negative definite**

Note that since F is affine, it takes necessarily the form $F(y) = F_0 + T(y)$, with $T\colon \mathbb{R}^N \to \mathbb{S}^{M\times M}$ a linear function

A non-strict LMI is a linear matrix inequality of the form $F(y) \preceq 0$

C. Scherer and S. Weiland, "Linear matrix inequalities in control". Notes for a course of the Dutch Institute of Systems and Control, 2004.

Functions of matrix variables as LMIs

LMIs often appear as **functions of matrix variables**, that is in the form

$$\hat{F}(Y) \prec 0 \qquad Y \in \mathbb{R}^{N_1 \times N_2} \text{ matrix variable}$$

where $\hat{F}(Y) = \hat{T}(Y) + \hat{F}_0$ and $\hat{T}(Y)$ linear.
This is a special case of $F(y) = F_0 + F_1 y_1 + \ldots + F_N y_N \prec 0$. Let $E_1, \ldots, E_n$ be a basis of $\mathbb{R}^{N_1 \times N_2}$ and let

$$Y = \sum_j y_j E_j, \quad y_j \in \mathbb{R}$$

Then

$$0 \succ \hat{F}(Y) = \hat{F}_0 + \hat{T}(\sum_j y_j E_j) = \underbrace{\hat{F}_0}_{=: F_0} + \sum_j y_j \underbrace{\hat{T}(E_j)}_{=: F_j}$$

Systems of LMIs

A system of LMIs

$$\begin{cases} F^{(1)}(y) \prec 0 \\ F^{(2)}(y) \prec 0 \\ \vdots \\ F^{(p)}(y) \prec 0 \end{cases}$$

is still an LMI, because it is equivalent to

$$\begin{bmatrix} F^{(1)}(y) & 0 & \dots & 0 \\ 0 & F^{(2)}(y) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & F^{(p)}(y) \end{bmatrix} \prec 0$$

which in turn is equivalent to

$$\begin{bmatrix} F_0^{(1)} & 0 & \dots & 0 \\ 0 & F_0^{(2)} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & F_0^{(p)} \end{bmatrix} + \sum_{j=1}^{N} y_j \begin{bmatrix} F_j^{(1)} & 0 & \dots & 0 \\ 0 & F_j^{(2)} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & F_j^{(p)} \end{bmatrix} \prec 0$$

Feasibility and optimization

LMI are studied in connection with the following two problems

- Feasibility whether or not there exists $y \in \mathbb{R}^N$ such that $F(y) \prec 0$
- Optimization Given a function $f\colon \mathcal{S} \to \mathbb{R}$, where $\mathcal{S} = \{y \in \mathbb{R}^N : F(y) \prec 0\}$, an optimization problem with LMI constraints is $\inf_{y\in\mathcal{S}} f(y)$.

An LMI defines a convex set, i.e., the set $\{y\colon F(y) \prec 0\}$ is a convex set, hence checking the feasibility of an LMI or optimizing a convex function over a constraint defined by an LMI is a **convex optimization problem**

Minimizing linear objective functions over symmetric semidefinite matrix variables belongs to the realm of semidefinite programming for which effective numerical methods and software are available.

Here to illustrate some examples we use CVX.

Schur complement

Schur complement is a powerful tool to linearize nonlinear inequalities.

Consider the LMI

$$F(y) = \begin{bmatrix} F_{11}(y) & F_{12}(y) \\ F_{21}(y) & F_{22}(y) \end{bmatrix} \prec 0$$

where $F_{11}(y)$, $F_{22}(x)$, $F_{12}(y)$ are affine functions. Then*

$$F(y) \prec 0$$
$$\Updownarrow$$
$$\begin{cases} F_{11}(y) \prec 0 \\ F_{22}(y) - F_{21}(y)[F_{11}(y)]^{-1}F_{12}(y) \prec 0 \end{cases}$$
$$\Updownarrow$$
$$\begin{cases} F_{22}(y) \prec 0 \\ F_{11}(y) - F_{12}(y)[F_{22}(y)]^{-1}F_{21}(y) \prec 0 \end{cases}$$

*The proof is based on the factorizations

$$\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} = \begin{cases} \begin{bmatrix} I & 0 \\ F_{21}F_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} - F_{21}F_{11}^{-1}F_{12} \end{bmatrix} \begin{bmatrix} I & F_{11}^{-1}F_{12} \\ 0 & I \end{bmatrix} & \text{if } F_{11} \text{ is invertible} \\ \begin{bmatrix} I & F_{12}F_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} F_{11} - F_{12}F_{22}^{-1}F_{21} & 0 \\ 0 & F_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ F_{22}^{-1}F_{21} & I \end{bmatrix} & \text{if } F_{22} \text{ is invertible} \end{cases}$$

Direct data-driven stabilization

Problem (Quadratic stabilization) Based on the dataset $\mathbb{D}$

$$\begin{aligned} \text{find} \quad & K, P \succ 0 \\ \text{such that} \quad & (A+BK)P(A+BK)^\top - P \prec 0 \end{aligned}$$

- If the quadratic stabilization problem is solvable, then $u = Kx$ makes the origin a globally exponentially stable equilibrium for the closed-loop system $x^+ = (A+BK)x$, i.e., $\exists c, 0 \le \rho < 1$ such that $|x(k)| \le c\rho^k |x(0)|$ for all $k \ge 0$ and all $x(0) \in \mathbb{R}^n$.
- $V(x) = x^\top P x$ is a strict Lyapunov function, i.e., a positive definite function such that $V(x^+) - V(x) < 0$ for all $x \neq 0$.

As A, B are unknown, to find a solution to the problem the idea is to work with $X_1 G_K$ instead of $A + BK$ under the condition for which $X_1 G_K = A + BK$

A formula for direct data-driven stabilization

For any K, G_K such that $\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G_K$, we have $A + BK = X_1 G_K$

Theorem Consider a system $x^+ = Ax + Bu$ with dataset U_0, X_1, X_0. Consider the decision variables

$$P \in \mathbb{R}^{n\times n},\ Y \in \mathbb{R}^{T\times n}$$

and the following SDP

$$X_0 Y = P \tag{1a}$$

$$\begin{bmatrix} -P & X_1 Y \\ Y^\top X_1^\top & -P \end{bmatrix} \prec 0 \tag{1b}$$

If it is feasible then the control gain

$$K = U_0 Y P^{-1}$$

solves the quadratic stabilization problem.

Let (1) be feasible. Constraint (1b) guarantees $P \succ 0$. Hence P is invertible. Constraint (1a) can be equivalently written as

$$\text{(1a) } X_0 Y = P \Leftrightarrow X_0 Y P^{-1} = I_n,$$

Perform the change of variable $G_K := Y P^{-1}$, to obtain $X_0 G_K = I_n$.

By the same change of variable, the control gain

$$K = U_0 Y P^{-1}$$

can be written as $K = U_0 G_K$

Hence, $\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G_K$. This returns the data-dependent representation of the closed-loop system, i.e., $A + BK = X_1 G_K$.

Consider constraint (1b) $\begin{bmatrix} -P & X_1 Y \\ Y^\top X_1^\top & -P \end{bmatrix} \prec 0$. By Schur complement, the inequality is equivalent to $P \succ 0$ and $-P + X_1 Y P^{-1} Y^\top X_1^\top \prec 0$. Rewrite the last inequality as $-P + X_1 Y P^{-1} P P^{-1} Y^\top X_1^\top \prec 0$. Bearing in mind the change of variable $G_K = Y P^{-1}$, the latter can be written as $-P + X_1 G_K P G_K^\top X_1^\top \prec 0$, or, by the identity $A + BK = X_1 G_K$, as

$$P \succ 0, \quad (A + BK) P (A + BK)^\top - P \prec 0$$

Data-based parameterization of all stabilizing controllers

Under the assumption of sufficiently rich data, i.e., $\begin{bmatrix} U_0 \\ X_0 \end{bmatrix}$ has full row rank, then one can parametrize via data all the controllers that solve the quadratic stabilization problem.

Corollary Assume that $\begin{bmatrix} U_0 \\ X_0 \end{bmatrix}$ has full row rank. Any control gain $K \in \mathbb{R}^{m \times n}$ that solves the quadratic stabilization problem must be of the form

$$K = U_0 Y P^{-1}$$

where Y, P are a solution of

$$X_0 Y = P \tag{2a}$$

$$\begin{bmatrix} -P & X_1 Y \\ Y^\top X_1^\top & -P \end{bmatrix} \prec 0 \tag{2b}$$

As K is stabilizing, $A+BK$ is Schur stable, that is, equivalently, there exists $P \succ 0$ such that $(A+BK)P(A+BK)^\top - P \prec 0$.

As $\left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right]$ has full row rank, by Rouché-Capelli theorem there must exist G_K such that

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G_K$$

Hence, $K = U_0 G_K$, $I_n = X_0 G_K$ and $A + BK = X_1 G_K$. The latter implies that the Lyapunov inequality can be equivalently rewritten as

$$P \succ 0, \quad X_1 G_K P (X_1 G_K)^\top - P \prec 0$$

Proceedings as before, one performs the change of variable $Y := G_K P$ and the Lyapunov inequality above is equivalently rewritten as

$$\begin{bmatrix} -P & X_1 Y \\ Y^\top X_1^\top & -P \end{bmatrix} \prec 0$$

The identities $K = U_0 G_K$, $I_n = X_0 G_K$ expressed in the variables Y, P return $K = U_0 Y P^{-1}$, $I_n = X_0 Y P^{-1}$.

Exercise

Consider the decision variables $P \in \mathbb{R}^{n\times n}, Y \in \mathbb{R}^{T\times n}$ and the following SDP

$$X_0 Y = P$$
$$\begin{bmatrix} -P & Y^\top X_1^\top \\ X_1 Y & -P \end{bmatrix} \prec 0 \tag{3a}$$

Show that, if it is feasible, then the control law $u = Kx$ with $K = U_0 Y P^{-1}$ and the matrix $P^{-1} \succ 0$ satisfy

$$(A+BK)^\top P^{-1}(A+BK) - P^{-1} \prec 0$$

Solution The proof proceeds as in the case of the previous result until the manipulation of the constraint (3a), which is goes in a slightly different way as follows. By Schur complement, the inequality is equivalent to $P \succ 0$ and $-P + Y^\top X_1^\top P^{-1} X_1 Y \prec 0$. Multiply the last inequality by P^{-1} on both sides, to obtain $-P^{-1} + P^{-1} Y^\top X_1^\top P^{-1} X_1 Y P^{-1} \prec 0$. Bearing in mind the change of variable $G = YP^{-1}$, the latter can be written as $-P^{-1} + G^\top X_1^\top P^{-1} X_1 G \prec 0$, or, by the identity $A + BK = X_1 G$, as $P \succ 0$, $(A+BK)^\top P^{-1}(A+BK) - P^{-1} \prec 0$, as claimed.

Example (cont'd)

Data-based stabilization of the unknown dynamics

State response to PE input from experiment

$$X_0 = \begin{bmatrix} 0.4027 & 0.3478 & 0.3571 & 0.3216 & 0.2362 \\ 0.4448 & 1.1451 & 1.7499 & 2.3708 & 2.9301 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 0.3478 & 0.3571 & 0.3216 & 0.2362 & 0.1541 \\ 1.1451 & 1.7499 & 2.3708 & 2.9301 & 3.3409 \end{bmatrix}$$

Solve for Y the (nonstrict) LMI

```
cvx_begin sdp
variable Y(T,n)
variable P(T,n) symmetric
[P X1*Y; Y'*X1' P]>=eye(2*n);
P=X0*Y
cvx_end
```

which returns

$$Y = \begin{bmatrix} 27.4724 & -20.8515 \\ -25.5235 & -8.8555 \\ -1.6399 & -2.0356 \\ 5.3938 & 3.6399 \\ 0.1696 & 18.8019 \end{bmatrix}$$

$$P = \begin{bmatrix} 3.3752 & -6.5922 \\ -6.5922 & 40.7437 \end{bmatrix}$$

S. Boyd. "Solving semidefinite programs using cvx," `http://stanford.edu/class/ee363/notes/lmi-cvx.pdf`

Example

Feedback gain

$$K = U_0 Y P^{-1} = \begin{bmatrix} -8.2995 & -1.2512 \end{bmatrix}$$

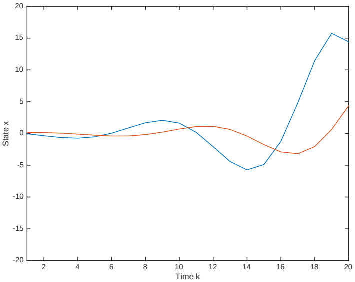


Figure: Unforced solution $u(k) = 0$

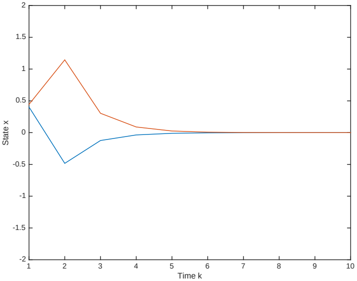


Figure: Solution under data-based feedback $u(k) = Kx(k)$

Spectral radius data-based controlled system $\rho(X_1 Y P^{-1}) = 0.5666$

A few comments

- Simple solution: data-dependent Lyapunov stability theory
- The data-based problem is solvable via efficient numerical algorithms (cvx)
- It only requires a finite number of data collected in one-shot low sample-complexity experiments ($T \geq (m+1)(n+1) - 1$)
- If the system is high-dimensional and highly unstable, then collecting data in one-shot experiment of length T might not be viable and one can use multiple dataset of shorter length
- The result provides a parametrization of all stabilizing state feedback gains
- The result can be extended to design dynamic output feedback control from data

The case of continuous-time systems

Input and state sampled trajectories Given a sampling time $\Delta > 0$, let

$$\begin{aligned} U_0 &= \begin{bmatrix} u_d(0) & u_d(\Delta) & \dots & u_d((T-1)\Delta) \end{bmatrix} \\ X_0 &= \begin{bmatrix} x_d(0) & x_d(\Delta) & \dots & x_d((T-1)\Delta) \end{bmatrix} \end{aligned}$$

Data-dependent representation of the closed-loop system As in the discrete-time case, $A + BK = X_1 G_K$ where

$$X_1 := \begin{bmatrix} \dot{x}_d(0) & \dot{x}_d(\Delta) & \dots & \dot{x}_d((T-1)\Delta) \end{bmatrix}$$

Lyapunov stability condition Any matrix Y satisfying

$$\begin{cases} X_1 Y + Y^\top X_1^\top \prec 0 \\ P = X_0 Y \succ 0 \end{cases}$$

is such that $K = U_0 Y (X_0 Y)^{-1}$ is a stabilizing feedback gain for the continuous-time system

Main difference Derivatives of the state at the sampling times X_1 are required $\Longrightarrow$ Noisy data (Lecture 2-4)

The case of continuous-time systems

Alternative[2] Integral version of $\dot{x} = Ax + Bu$

$$\overbrace{x((k+1)T_s) - x(kT_s)}^{\xi(k)} = A \overbrace{\int_{kT_s}^{(k+1)T_s} x(t)dt}^{r(k)} + B \overbrace{\int_{kT_s}^{(k+1)T_s} u(t)dt}^{v(k)}$$

and work with the relation

$$\overbrace{\left[\xi(0)\ldots\xi(T-1)\right]}^{\underline{X}_1} = A \overbrace{\left[r(0)\ldots r(T-1)\right]}^{\underline{X}_0} + B \overbrace{\left[v(0)\ldots v(T-1)\right]}^{\underline{U}_0}$$

Lyapunov stability condition Any matrix Y satisfying

$$\begin{cases} X_1 Y + Y^\top X_1^\top \prec 0 \\ P = X_0 Y \succ 0 \end{cases}$$

is such that $K = U_0 Y (X_0 Y)^{-1}$ is a stabilizing feedback gain for the continuous-time system (and does not require state derivatives!)

[2] De Persis, Postoyan, Tesi. Event-triggered control from data. IEEE Transactions on Automatic Control (provisionally accepted), arXiv:2208.11634

Optimality

Optimality - Linear Quadratic Regulation

LQR problem Assume (A, B) reachable. Consider the problem of minimizing

$$J_\infty(x_0, u) := \sum_{k=0}^{\infty} (x(k)Qx(k) + u(k)^\top Ru(k)), \quad Q \succ 0, R \succ 0$$

over the set of input sequences $u\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}^m$ for which the solution $x\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}^n$ to $x(k+1) = Ax(k) + Bu(k)$, $x(0) = x_0$, satisfies $\lim_{k\to\infty} x(k) = 0$.

There exists a unique optimal controller given by

$$u_\star := K_\star x, \quad K_\star := -(R + B^\top PB)^{-1}B^\top PA$$

where $P \succ 0$ is the unique solution of the DARE

$$A^\top PA - P - A^\top PB(R + B^\top PB)^{-1}B^\top PA + Q = 0$$

that renders the matrix $A - B(R + B^\top PB)^{-1}B^\top PA$ Schur stable. Moreover, the optimal cost is $x_0^\top Px_0$.

Importance of data-driven LQR

- Infinite-horizon LQR is the prime example of challenges encountered in data-driven optimal control (effect of noise, deviation from optimality)
- Of interest to both the data-driven control and machine learning community

A reformulation of LQR: computing $K_\star$ via SDP

For the system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + \xi(k) \\ z(k) &= \begin{bmatrix} Q^{1/2} & 0 \\ 0 & R^{1/2} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \end{aligned} \qquad (\star)$$

design K that

- makes $A + BK$ Schur stable
- minimizes the sum of the squares of the energy of the output responses to the impulse inputs of the closed-loop system

$$\begin{aligned} x(k+1) &= (A+BK)x(k) + \xi(k) \\ z(k) &= \begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix} x(k) \end{aligned}$$

Impulse response

Consider the Schur stable closed-loop system

$$\begin{aligned} x(k+1) &= \underbrace{(A+BK)}_{A_c} x(k) + \underbrace{I_n}_{B_c} \cdot \xi(k) \\ z(k) &= \underbrace{\begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix}}_{C_c} x(k) \end{aligned}$$

and compute the output energy of the impulse responses of the system.

- Let $z^{(j)}$ be the response to the impulse $e_j \delta(k)$, with e_j the j-th vector of the canonical basis of $\mathbb{R}^n$ and $\delta(k)$ the discrete-time impulse

$$z^{(j)}(k) = \begin{cases} 0 & k = 0 \\ C_c A_c^{k-1} e_j & k > 0 \end{cases}$$

- Let $\|z^{(j)}\|_2^2$ denote its energy (the series is summable because A_c is Schur)

$$\sum_{k=0}^{\infty} \|z^{(j)}(k)\|^2 = \sum_{k=0}^{\infty} e_j^\top (A_c^\top)^k C_c^\top C_c A_c^k e_j = \sum_{k=0}^{\infty} \operatorname{trace}(C_c A_c^k e_j e_j^\top (A_c^\top)^k C_c^\top)$$

Then

$$\begin{aligned} \sum_{j=1}^{n} \|z^{(j)}\|_2^2 &= \operatorname{trace}\Big(\sum_{k=0}^{\infty} C_c A_c^k B_c B_c^\top (A_c^\top)^k C_c^\top\Big) \\ &= \operatorname{trace}\Big(\sum_{k=0}^{\infty} B_c^\top (A_c^\top)^k C_c^\top C_c A_c^k B_c\Big) \end{aligned}$$

From $\sum_{j=1}^n \|z^{(j)}\|_2^2 = \operatorname{trace}\Big(\sum_{k=0}^{\infty} C_c A_c^k B_c B_c^\top (A_c^\top)^k\Big) = \operatorname{trace}\Big(C_c\Big(\sum_{k=0}^{\infty} A_c^k B_c B_c^\top (A_c^\top)^k\Big)C_c^\top\Big)$,

if one sets

$$P := \sum_{k=0}^{\infty} A_c^k B_c B_c^\top (A_c^\top)^k$$

one realizes that P, the <u>controllability gramian</u>, is the (unique) positive semidefinite matrix satisfying

$$A_c P A_c^\top - P + B_c B_c^\top = (A+BK)P(A+BK)^\top - P + I = 0$$

The last equation and $P \succeq 0$ implies that

$$P = (A+BK)P(A+BK)^\top + I \succeq I$$

Finally

$$\begin{aligned}\sum_{j=1}^n \|z^{(j)}\|_2^2 = & \quad \operatorname{trace}\Big(C_c P C_c^\top\Big) \\ = & \quad \operatorname{trace}\Big(\begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix} P \begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix}^\top\Big) = \operatorname{trace}(QP) + \operatorname{trace}(R^{1/2}KPK^\top R^{1/2})\end{aligned}$$

In summary The sum of the squares of the energy of the output responses to the impulse inputs of the Schur stable system

$$
\begin{aligned}
x(k+1) &= \underbrace{(A+BK)}_{A_c} x(k) + \underbrace{I_n}_{B_c} \cdot \xi(k) \\
z(k) &= \underbrace{\begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix}}_{C_c} x(k)
\end{aligned}
$$

is given by

$$\sum_{j=1}^{n} \|z^{(j)}\|_2^2 = \text{trace}(QP) + \text{trace}(R^{1/2}KPK^\top R^{1/2})$$

with P the unique matrix satisfying

$$
\begin{aligned}
&(A+BK)P(A+BK)^\top - P + I_n = 0 \\
&P \succeq I_n
\end{aligned}
$$

The $\mathcal{H}_2$-norm minimization problem

$\mathcal{H}_2$-norm By the discrete-time version of Parseval's theorem

$$\sum_{j=1}^{n} \|z^{(j)}\|_2^2 = \|\mathcal{T}(K)\|_2^2$$

where $\|\mathcal{T}(K)\|_2^2$ is the $\mathcal{H}_2$-norm* of the transfer function $\mathcal{T}(K)$ of the Schur stable system

$$x(k+1) = \underbrace{(A+BK)}_{A_c} x(k) + \underbrace{I_n}_{B_c} \cdot \xi(k)$$

$$z(k) = \underbrace{\begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix}}_{C_c} x(k)$$

$^\star \|\mathcal{T}(K)\|_2^2 := \frac{1}{2\pi}\int_0^{2\pi} \mathrm{trace}\Big(\mathcal{T}(\mathrm{e}^{i\theta})^* \mathcal{T}(\mathrm{e}^{i\theta})\Big) d\theta$ where $\mathcal{T}(\mathrm{e}^{i\theta}) := \mathcal{T}(K)|_{z=\mathrm{e}^{i\theta}}$

The state feedback controller that minimizes $\|\mathcal{T}(K)\|_2^2$, i.e., that solves

$$\begin{aligned} \min_{K,P} \quad & \mathrm{trace}(QP) + \mathrm{trace}(R^{1/2}KPK^\top R^{1/2}) \\ \text{subject to} \quad & \begin{cases} (A+BK)P(A+BK)^\top - P + I_n = 0 \\ P \succeq I_n \end{cases} \end{aligned}$$

is unique and coincides with the solution to the LQR problem, i.e., $K = K_\star$ (Chen-Francis, *Optimal sampled-data control system*, Section 6.4).

A semidefinite program for solving the $\mathcal{H}_2$-norm minimization problem

The previous arguments suggest the following convex relaxation of the $\mathcal{H}_2$-norm minimization problem

$$\min_{K,P,L}\ \operatorname{trace}(QP) + \operatorname{trace}(L)$$

subject to

$$\begin{cases} (A+BK)P(A+BK)^\top - P + I_n \preceq 0 \\ P \succeq I_n \\ L - R^{1/2}KPK^\top R^{1/2} \succeq 0 \end{cases}$$

where the equality constraint is relaxed to an inequality and the constraint

$$R^{1/2}KWK^\top R^{1/2} \preceq L$$

is introduced to remove

$$\operatorname{trace}(R^{1/2}KWK^\top R^{1/2})$$

from the cost function and replace it with the linear term $\operatorname{trace}(L)$.

By (Feron *et al.*, Proposition 1), under the given assumptions, the problem above is well-posed, ie. the feasible set is compact or empty. As the feasible set is non-empty, then the feasible set is compact.

E. Feron, V. Balakrishnan, S. Boyd, L. El Ghaoui, "Numerical methods for H_2 related problems," in 1992 American Control Conference, pp. 2921–2922.

A data-dependent solution to the LQR

The $\mathcal{H}_2$-norm minimization problem and its convex relaxation

$$
\begin{array}{ll}
\min_{K,P} & \mathrm{trace}(QP) + \mathrm{trace}(R^{1/2}KPK^\top R^{1/2}) \\
\text{subject to} & \begin{cases} (A+BK)P(A+BK)^\top - P + I_n = 0 \\ P \succeq I_n \end{cases}
\end{array}
\qquad
\begin{array}{ll}
\min_{K,P,L} & \mathrm{trace}\,(QP) + \mathrm{trace}\,(L) \\
\text{subject to} & \begin{cases} (A+BK)P(A+BK)^\top - P + I_n \preceq 0 \\ P \succeq I_n \\ L - R^{1/2}KPK^\top R^{1/2} \succeq 0 \end{cases}
\end{array}
$$

are related as follows

Proposition A solution $(\overline{K}, \overline{P}, \overline{L})$ to the convex relation is such that $(\overline{K}, \overline{P})$ is the solution to the $\mathcal{H}_2$-norm minimization problem. Moreover, $\overline{K} = K_\star$, that is, $\overline{K}$ is the solution to the optimal LQR problem.

A data-dependent solution to the LQR

The previous optimization problem leads to the following data-dependent SDP for designing the LQR from data

$$\begin{aligned}
&\min_{G,P,L} \ \operatorname{trace}(QP) + \operatorname{trace}(L) \\
&\text{subject to} \\
&\begin{cases}
X_1 G P G^\top X_1^\top - P + I_n \preceq 0 \\
P \succeq I_n \\
L - R^{1/2} U_0 G P G^\top U_0^\top R^{1/2} \succeq 0 \\
X_0 G = I_n
\end{cases}
\end{aligned} \qquad \text{(DD-SDP-LQR)}$$

Theorem Assume that $\begin{bmatrix} U_0 \\ X_0 \end{bmatrix}$ has full row rank. Any optimal solution (G^o, P^o, L^o) to (DD-SDP-LQR) is such that $K^o := U_0 G^o$ satisfies

$$K_\star = K^o$$

and

$$\|\mathcal{T}(K^o)\|_2^2 = \operatorname{trace}(QP^o) + \operatorname{trace}(L^o)$$

A sketch of proof

Lemma 1 Consider any control gain K stabilising for

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + \xi(k) \\ z(k) &= \begin{bmatrix} Q^{1/2} & 0 \\ 0 & R^{1/2} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \end{aligned} \qquad (\star)$$

Then there exists a triple (G_K, P, L) feasible for (DD-SDP-LQR) such that

$$K = U_0 G_K \quad \text{and} \quad \|\mathcal{T}(K)\|_2^2 = \operatorname{trace}(QP) + \operatorname{trace}(L)$$

A sketch of proof

Lemma 1 Consider any control gain K stabilising for $(\star)$. Then there exists a triple (G_K, P, L) feasible for (DD-SDP-LQR) such that

$$K = U_0 G_K \quad \text{and} \quad \|\mathcal{T}(K)\|_2^2 = \operatorname{trace}(QP) + \operatorname{trace}(L)$$

For a given K, let G_K be such that

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G_K \quad \iff \quad K = U_0 G_K, \; I_n = X_0 G_K$$

As K is stabilizing, $A + BK = X_1 G_K$ is Schur stable and there exists a unique controllability gramian P such that

$$X_1 G_K P X_1 G_K^\top - P + I = 0, \quad P \succeq I$$

Moreover, $\|\mathcal{T}(U_0 G_K)\|_2^2 = \operatorname{trace}(QP) + \operatorname{trace}(R^{1/2} U_0 G_K P G_K^\top U_0^\top R^{1/2})$

Set $L := R^{1/2} U_0 G_K P G_K^\top U_0^\top R^{1/2}$. Then

$$\|\mathcal{T}(U_0 G_K)\|_2^2 = \operatorname{trace}(QP) + \operatorname{trace}(L)$$

and (G_K, P, L) is feasible for (DD-SDP-LQR)

A sketch of proof

Lemma 1 Consider any control gain K stabilising for $(\star)$. Then there exists a triple (G_K, P, L) feasible for (DD-SDP-LQR) such that

$$K = U_0 G_K \quad \text{and} \quad \|\mathcal{T}(K)\|_2^2 = \text{trace}(QP) + \text{trace}(L)$$

The feasible solution (G_K, P, L) to

$$\begin{aligned} &\min_{G_K, P, L} \ \text{trace}\,(QP) + \text{trace}\,(L) \\ &\text{subject to} \\ &\begin{cases} X_1 G_K P G_K^\top X_1^\top - P + I_n \preceq 0 \\ P \succeq I_n \\ L - R^{1/2} U_0 G_K P G_K^\top U_0^\top R^{1/2} \succeq 0 \\ X_0 G_K = I_n \end{cases} \end{aligned} \qquad \text{(DD-SDP-LQR)}$$

was obtained by

- computing G_K as a solution to $\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G_K$
- Setting P equal to the controllability gramian, i.e. $X_1 G_K P G_K^\top X_1^\top - P + I_n = 0$, $P \succeq I_n$
- Setting $L = R^{1/2} U_0 G_K P G_K^\top U_0^\top R^{1/2}$

A sketch of proof

Lemma 2 Any feasible solution (G_K, P, L) to (DD-SDP-LQR) is such that $K = U_0 G_K$ is stabilizing for $(\star)$ and
$$\|\mathcal{T}(K)\|_2^2 \leq \mathrm{trace}(QP) + \mathrm{trace}(L)$$

Proof – see Exercise #1

A sketch of proof

Exercise #1

(a) Show that $K = U_0 G_K$ is stabilising.

As $I_n = X_0 G_K$, setting $K = U_0 G_K$ yields $A + BK = X_1 G_K$. Since (G_K, P, L) is a feasible solution, $P \succeq I$ and $X_1 G_K P X_1 G_K{}^\top - P + I \preceq 0$ show that $X_1 G_K$ is Schur stable, hence $K = U_0 G_K$ is stabilising.

(b) Show that the inequality $X_1 G_K P X_1 G_K{}^\top - P + I \preceq 0$ implies the existence of a matrix Θ such that P is the controllability Gramian of the system

$$\begin{aligned} x(k+1) &= \quad X_1 G_K x(k) + \begin{bmatrix} I & \Theta \end{bmatrix} \xi(k) \\ z(k) &= \quad \begin{bmatrix} Q^{1/2} \\ R^{1/2} K \end{bmatrix} x(k) \end{aligned}$$

Since $X_1 G_K$ is Schur stable, P is the controllability gramian for the system if and only if

$$X_1 G_K P X_1 G_K{}^\top - P + I + \begin{bmatrix} I & \Theta \end{bmatrix} \begin{bmatrix} I & \Theta \end{bmatrix}^\top = 0$$

Hence, one needs to prove the existence of a matrix Θ such that the equation above holds. Since $X_1 G_K P G_K{}^\top X_1^\top - P + I \preceq 0$, then there exists Θ such that

$$X_1 G_K P X_1 G_K{}^\top - P + I + \Theta\Theta^\top = 0$$

In fact, set $\Xi := -(X_1 G_K P X_1 G_K{}^\top - P + I)$. Then $X_1 G_K P X_1 G_K{}^\top - P + I + \Xi = 0$. Since $\Xi \succeq 0$, by Cholesky factorization, we have $\Xi = \Theta\Theta^\top$.

A sketch of proof

(c) Show that $\|\mathcal{T}_e(K)\|_2^2 = \text{trace}(QP) + \text{trace}(R^{1/2}KPK^\top R^{1/2})$, where $\mathcal{T}_e(K)$ is the transfer function of

$$\begin{aligned} x(k+1) = &\quad X_1 G_K x(k) + \begin{bmatrix} I & \Theta \end{bmatrix} \xi(k) \\ z(k) = &\quad \begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix} x(k) \end{aligned}$$

Since P is the controllability gramian for the system, then

$$\|\mathcal{T}_e(K)\|_2^2 = \text{trace}\Big(\begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix} P \begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix}^\top\Big)$$

and the claim follows immediately by the definition of trace.

(d) Conclude $\|\mathcal{T}(K)\|_2^2 \leq \|\mathcal{T}_e(K)\|_2^2 \leq \text{trace}(QP) + \text{trace}(L)$

By Parseval's theorem the total energy of the output impulsive responses equals the $\mathcal{H}_2$-norm squared of the system

$$\|\mathcal{T}_e(K)\|_2^2 = \text{trace}\Big(\sum_{k=0}^{\infty} C_c \cdot (X_1 G_K)^k \begin{bmatrix} I & \Theta \end{bmatrix} \begin{bmatrix} I \\ \Theta^\top \end{bmatrix} (G_K^\top X_1^\top)^k C_c^\top\Big)$$

A sketch of proof

Hence

$$
\begin{aligned}
&\|\mathcal{T}_e(K)\|_2^2 = \mathrm{trace}\Big(\sum_{k=0}^{\infty} C_c \cdot (X_1 G_K)^k \begin{bmatrix} I & \Theta \end{bmatrix} \begin{bmatrix} I \\ \Theta^\top \end{bmatrix} (G_K^\top X_1^\top)^k C_c^\top \Big) = \\
&\mathrm{trace}\Big(\sum_{k=0}^{\infty} C_c (X_1 G_K)^k\, I\, (G_K^\top X_1^\top)^k C_c^\top \Big) + \mathrm{trace}\Big(\sum_{k=0}^{\infty} C_c (X_1 G_K)^k \Theta\Theta^\top (G_K^\top X_1^\top)^k C_c^\top \Big) \geq \\
&\mathrm{trace}\Big(\sum_{k=0}^{\infty} C_c (X_1 G_K)^k\, I\, (G_K^\top X_1^\top)^k C_c^\top \Big) = \|\mathcal{T}(K)\|_2^2
\end{aligned}
$$

The claim follows since

$$
\begin{aligned}
\|\mathcal{T}_e(K)\|_2^2 &\overset{(c)}{=} \mathrm{trace}(QP) + \mathrm{trace}(R^{1/2} K P K^\top R^{1/2}) \\
&\overset{R^{1/2} K P K^\top R^{1/2} \preceq L}{\leq} \mathrm{trace}(QP) + \mathrm{trace}(L)
\end{aligned}
$$

A sketch of proof – final argument

An optimal solution (G_K^o, P^o, L^o) to (DD-SDP-LQR) satisfies (Lemma 2)

$$\|\mathcal{T}(K^o)\|_2^2 \leq \text{trace}(QP^o) + \text{trace}(L^o) \quad \text{with} \quad K^o := U_0 G_K^o$$

On the other hand, since $K_\star$ is stabilizing, there exists a feasible $(G_{K_\star}, P_\star, L_\star)$ for (DD-SDP-LQR) such that (Lemma 1)

$$K_\star = U_0 G_{K_\star} \quad \text{and} \quad \|\mathcal{T}(K_\star)\|_2^2 = \text{trace}(QP_\star) + \text{trace}(L_\star)$$

As (G_K^o, P^o, L^o) is an optimal solution to (DD-SDP-LQR), it is true that

$$\text{trace}(QP^o) + \text{trace}(L^o) \leq \text{trace}(QP_\star) + \text{trace}(L_\star)$$

which implies

$$\|\mathcal{T}(K^o)\|_2^2 \leq \text{trace}(QP^o) + \text{trace}(L^o) \leq \text{trace}(QP_\star) + \text{trace}(L_\star) = \|\mathcal{T}(K_\star)\|_2^2$$

As $K_\star$ is the optimal solution to the $\mathcal{H}_2$-norm minimization problem, $\|\mathcal{T}(K_\star)\|_2^2 \leq \|\mathcal{T}(K^o)\|_2^2$, that is $\|\mathcal{T}(K_\star)\|_2^2 = \|\mathcal{T}(K^o)\|_2^2$ and by uniqueness of the optimal gain, $K^o = K_\star$

A data-dependent solution to the LQR

Recap We have shown the correctness of the following data-dependent SDP for designing the LQR from data

$$
\begin{aligned}
&\min_{G_K,P,L} \ \operatorname{trace}(QP) + \operatorname{trace}(L) \\
&\text{subject to} \\
&\begin{cases}
X_1 G_K P G_K^\top X_1^\top - P + I_n \preceq 0 \\
P \succeq I_n \\
L - R^{1/2} U_0 G_K P G_K^\top U_0^\top R^{1/2} \succeq 0 \\
X_0 G_K = I_n
\end{cases}
\end{aligned}
\qquad \text{(DD-SDP-LQR)}
$$

Theorem Assume that $\left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right]$ has full row rank. Any optimal solution (G_K^o, P^o, L^o) to (DD-SDP-LQR) is such that $K^o := U_0 G_K^o$ satisfies

$$K_\star = K^o$$

and

$$\|\mathcal{T}(K^o)\|_2^2 = \operatorname{trace}(QP^o) + \operatorname{trace}(L^o)$$

A data-dependent SDP for solving the LQR

The change of variables $Y = G_K P$ and an application of Schur complement lead to the semidefinite program

$$\min_{Y,P,L} \ \operatorname{trace}(QP) + \operatorname{trace}(L)$$

subject to

$$\begin{cases} \begin{bmatrix} P - I_n & X_1 Y \\ Y^\top X_1^\top & P \end{bmatrix} \succeq 0 \\ \begin{bmatrix} L & R^{1/2} U_0 Y \\ Y^\top U_0^\top R^{1/2} & P \end{bmatrix} \succeq 0 \\ \\ P = X_0 Y \end{cases}$$

with the optimal gain matrix given by

$$K_\star = U_0 Y P^{-1}$$

C. De Persis, P. Tesi. “Formulas for data-driven control: stabilization, optimality, robustness”. IEEE Transactions on Automatic Control, 65(3), 909-924, 2020.

Discussion

- The data-based problem is solvable via efficient numerical algorithms (cvx)

```
cvx_begin sdp
    variable Y(T,n)
    variable L(m,m) symmetric
    variable P(n,n) symmetric
    minimize ( trace(Q*P) +trace(L) )
    [L, sqrtm(R)*U0*Y; Y'*U0'*sqrtm(R)', P] >= 0
    [P-eye(n), X1*Y; Y'*X1', P] >= 0
    P=X0*Y
cvx_end
K = U0*Y*inv(P);
```

- It only requires data collected in low sample-complexity experiments
- Solution is exactly computed via a single SDP and not approximated via sequential iterations as in, e.g., LQR via policy iteration

Policy iteration and LQR

Algorithm 1 Policy iteration applied to the LQR problem

```
Guess initial stabilizing gain K_0
Set initial time k = 0
for i = 0 to ∞ do
    for j = 1 to N do
        Apply u(k) = K_i x(k) + e(k), e(k) PE "exploration signal"
        Estimate K_i(j) using RLS and I/O measurements
        k = k + 1
    end for
    Set K_{i+1} = K_i(N)
end for
```

There exists an estimation interval N such that the algorithm generates a sequence $\{K_i \, : \, i = 0, 1, 2, \ldots\}$ such that $\lim_{i\to\infty} \|K_i - K_\star\| = 0$

S.J. Bradtke, B.E. Ydstie and A.G. Barto. Adaptive linear quadratic control using policy iteration. Proceedings of the 1994 American Control Conference, 3475–3479, 1994.

The data-dependent solution to LQR with noisy data

$$\min_{G_K,P,L} \ \operatorname{trace}(QP) + \operatorname{trace}(L)$$

subject to

$$\begin{cases} X_1 G_K P G_K^\top X_1^\top - P + I_n \preceq 0 \\ P \succeq I_n \\ L - R^{1/2} U_0 G_K P G_K^\top U_0^\top R^{1/2} \succeq 0 \\ X_0 G_K = I_n \end{cases}$$

As any other result in this Lecture 1, this program is derived from noise-free data

In the presence of noise, brought in by the unknown matrix D_0 (Lecture 2), the data-dependent representation leads to the SDP $\Rightarrow$

$$\min_{G_K,P,L} \ \operatorname{trace}(QP) + \operatorname{trace}(L)$$

subject to

$$\begin{cases} (X_1 - D_0) G_K P G_K^\top (X_1 - D_0)^\top - P + I_n \preceq 0 \\ P \succeq I_n \\ L - R^{1/2} U_0 G_K P G_K^\top R^{1/2} \succeq 0 \\ X_0 G_K = I_n \end{cases}$$

The resulting optimal gain matrix is $K^o = U_0 Y P^{-1}$, which coincides with $K_\star$

The data-dependent solution to LQR with noisy data

$$
\begin{aligned}
&\min_{G_K,P,L}\ \operatorname{trace}(QP)+\operatorname{trace}(L)\\
&\text{subject to}\\
&\begin{cases}
X_1 G_K P G_K^\top X_1^\top - P + I_n \preceq 0\\
P \succeq I_n\\
L - R^{1/2} U_0 G_K P G_K^\top U_0^\top R^{1/2} \succeq 0\\
X_0 G_K = I_n
\end{cases}
\end{aligned}
$$

As any other result in this Lecture 1, this program is derived from noise-free data

In the presence of noise, brought in by the unknown matrix D_0 (Lecture 2), the data-dependent representation leads to the SDP $\Rightarrow$

$$
\begin{aligned}
&\min_{G_K,P,L}\ \operatorname{trace}(QP)+\operatorname{trace}(L)\\
&\text{subject to}\\
&\begin{cases}
(X_1 - D_0) G_K P G_K^\top (X_1 - D_0)^\top - P + I_n \preceq 0\\
P \succeq I_n\\
L - R^{1/2} U_0 G_K P G_K^\top R^{1/2} \succeq 0\\
X_0 G_K = I_n
\end{cases}
\end{aligned}
$$

The resulting optimal gain matrix is $K^o = U_0 Y P^{-1}$, which coincides with $K_\star$

Data-dependent solution to LQR - Soft constraint

- Since D_0 is unknown, one option is to neglect D_0 and require the term $M = G_K P G_K^\top$ to be small via the hard constraint $2M \preceq \epsilon I$
- The hard constraint, however, favours too much robustness to the detriment of performance

We instead look for a solution that trades off robustness for performance via a soft constraint

$$\min_{Y,P,L,V} \ \operatorname{trace}(QP) + \operatorname{trace}(L) + \alpha \operatorname{trace}(V)$$

subject to

$$\begin{cases} X_1 G_K P G_K^\top X_1^\top - P + I_n \preceq 0 \\ P \succeq I_n \\ L - R^{1/2} U_0 G_K P G_K^\top U_0^\top R^{1/2} \succeq 0 \\ V - G_K P G_K^\top \succeq 0 \\ X_0 G_K = I_n \end{cases}$$

where

$\alpha \gg 1$ favours robustness

$\alpha \ll 1$ favours performance

C. De Persis, P. Tesi. “Low-complexity learning of Linear Quadratic Regulators from noisy data”. Automatica 128, 109548, 2021

Partial information

Output feedback stabilization problem

Minimal SISO space representation with output measurements, with $\underline{A, B, C \text{ unknown matrices}}$

$$\begin{aligned} x(k+1) &= \quad Ax(k) + Bu(k) \quad && x(k) \in \mathbb{R}^n, \ u(k) \in \mathbb{R} \\ y(k) &= \quad Cx(k) && y(k) \in \mathbb{R}, \ k = 0, 1, 2, \ldots \end{aligned}$$

Design <u>from data</u> a dynamic output feedback controller

$$\begin{aligned} z^c(k+1) &= \quad Fz^c(k) + Gy(k) \\ u(k) &= \quad Hz^c(k) \end{aligned}$$

such that the closed-loop system

$$\begin{bmatrix} x(k+1) \\ z^c(k+1) \end{bmatrix} = \begin{bmatrix} A & BH \\ GC & F \end{bmatrix} \begin{bmatrix} x(k) \\ z^c(k) \end{bmatrix}$$

is asymptotically stable.

Output feedback stabilization problem - rationale

Minimal SISO space representation with output measurements

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) & x(k) \in \mathbb{R}^n,\ u(k) \in \mathbb{R} \\ y(k) &= Cx(k) & y(k) \in \mathbb{R},\ k = 0, 1, 2, \dots \end{aligned}$$

<u>Rationale</u> Reduce the data-driven output feedback control design to the state feedback one

- At a time k, we build a vector $\phi(k)$ of the past n values of the input and output samples

$$\phi(k) = \begin{bmatrix} y(k-n) & \dots & y(k-1) & u(k-n) & \dots & u(k-1) \end{bmatrix}^\top$$

<u>Lemma 3</u> There exist matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}$, depending on A, B, C, and where $\mathcal{A}, \mathcal{B}$ is a <u>reachable pair</u>, that describes the dynamics of ϕ

$$\begin{aligned} \phi(k+1) &= \mathcal{A}\phi(k) + \mathcal{B}u(k) \\ y(k) &= \mathcal{C}\phi(k), & k \geq n \end{aligned}$$

Observe that <u>$\phi(k)$ is a measured vector</u>, whereas <u>$\mathcal{A}, \mathcal{B}$ are unknown</u>

The result means the following:

▷ Denote by $\{\overline{u}(k)\colon k \geq 0\}, \{\overline{y}(k)\colon k \geq 0\}$ an I/O sequence generated by

$$\begin{aligned} x(k+1) &= \quad Ax(k) + Bu(k) \\ y(k) &= \quad Cx(k), \qquad\qquad k \geq 0 \end{aligned}$$

▷ Denote by $\{\tilde{u}(k)\colon k \geq n\}, \{\tilde{y}(k)\colon k \geq 0\}$ an I/O sequence generated by

$$\begin{aligned} \phi(k+1) &= \quad \mathcal{A}\phi(k) + \mathcal{B}u(k) \\ y(k) &= \quad \mathcal{C}\phi(k) \qquad\qquad k \geq n \end{aligned}$$

There exists an initial condition $\phi(n)$ such that, if $\tilde{u}(k) = u(k)$ for all $k \geq n$, then $\tilde{y}(k) = y(k)$ for all $k \geq n$.

Exercise #2

Exercise #2 – Prove Lemma 3

(a) Use the observable canonical form for $x(k+1) = Ax(k) + Bu(k)$, $y(k) = Cx(k)$ to show that the matrices $(\mathcal{A}, \mathcal{B})$ in $\phi(k+1) = \mathcal{A}\phi(k) + \mathcal{B}u(k)$ take the form

$$\mathcal{A} = \left[\begin{array}{ccccc|ccccc} 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n & b_1 & b_2 & b_3 & \cdots & b_n \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{array}\right] \quad \mathcal{B} = \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{array}\right]$$

for some parameters a's and b's.

(b) Use the Key Reachability Lemma ("*The pair* $(\mathcal{A}, \mathcal{B})$ *above is reachable if and only the polynomials* $z^n + a_n z^{n-1} \ldots + a_2 z + a_1$, $b_n z^{n-1} + \ldots + b_2 z + b_1$ *are coprime*") to conclude that $(\mathcal{A}, \mathcal{B})$ is reachable.

G.C. Goodwin, K.S. Sin. Adaptive Filtering Prediction and Control. Courier Corporation, 2014.

Exercise #2 – answers

Answers

(a) The system $x(k+1) = Ax(k) + Bu(k)$, $y(k) = Cx(k)$ is observable, hence there exists the observable canonical form

$$z(k+1) = \begin{bmatrix} -a_n & 1 & 0 & \cdots & 0 \\ -a_{n-1} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_2 & 0 & 0 & \cdots & 1 \\ -a_1 & 0 & 0 & \cdots & 0 \end{bmatrix} z(k) + \begin{bmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_2 \\ b_1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} z(k)$$

where $z^n + a_n z^{n-1} \ldots + a_2 z + a_1$ is the characteristic polynomial of the matrix A and $b_1, b_2, \ldots, b_n$ are coefficients depending on the matrices A, B, C. From the observable canonical form, we deduce the following left difference operator (a Deterministic Auto Regressive Moving Average) representation

$$\begin{aligned} y(k) + a_n y(k-1) + \ldots + a_2 y(k-n+1) + a_1 y(k-n) \\ = b_n u(k-1) + \ldots + b_2 u(k-n+1) + b_1 u(k-n) \end{aligned}$$

Exercise #2 – answers

Answers

From this representation it is straightforward to infer the pair of matrices $(\mathcal{A}, \mathcal{B})$ given in the statement:

$$
\begin{aligned}
\phi_1(k+1) = \quad & y(k-n+1) = \phi_2(k) \\
\vdots \quad & \\
\phi_n(k+1) = \quad & y(k) = -a_n y(k-1) \ldots - a_2 y(k-n+1) - a_1 y(k-n) \\
& \qquad +b_n u(k-1) + \ldots + b_2 u(k-n+1) + b_1 u(k-n) \\
\phi_{n+1}(k+1) = \quad & u(k-n+1) = \phi_{n+2}(k) \\
\vdots \quad & \\
\phi_{2n}(k+1) = \quad & u(k)
\end{aligned}
$$

(b) It is enough to observe that, since (A, B, C) is minimal, the polynomials $z^n + a_n z^{n-1} \ldots + a_2 z + a_1$, $b_n z^{n-1} + \ldots + b_2 z + b_1$ are coprime (they coincide with the numerator and denominator of the transfer function $C(zI - A)^{-1}B$ and no pole/zero cancellations are possible)

Collecting input-output data

Minimal SISO space representation with output measurements

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) & x(k) \in \mathbb{R}^n,\ u(k) \in \mathbb{R} \\ y(k) &= Cx(k) & y(k) \in \mathbb{R},\ k = 0, 1, 2, \ldots \end{aligned}$$

Experiment

- Consider an input $u_{[-n,T-1]}$, with $u_{[0,T-1]}$ PE of order $2n+1$, $T \geq (m+1)L - 1 = 2(2n+1) - 1$
- At time $k = -n$ from the initial condition $x(-n)$, apply $u_{[-n,T-1]}$ to the system and collect the **measured output** response in the $(2n+1) \times T$ matrix

$$\left[\begin{array}{l|l|l|l} u(0) & u(1) & \ldots & u(T-1) \\ \hline y(-n) & y(-n+1) & \ldots & y(-n+T-1) \\ y(-n+1) & y(-n+2) & \ldots & y(T-1) \\ \vdots & \vdots & \ddots & \vdots \\ y(-1) & u(0) & \ldots & y(T-2) \\ u(-n) & u(-n+1) & \ldots & u(-n+T-1) \\ u(-n+1) & u(-n+2) & \ldots & u(T-1) \\ \vdots & \vdots & \ddots & \vdots \\ u(-1) & u(0) & \ldots & u(T-2) \end{array}\right]$$

Collecting input-output data

Experiment (cont'd)

- The <u>input-output</u> matrix of data collected from system (C, A, B)

$$\left[\begin{array}{c|c|c|c} u(0) & u(1) & \dots & u(T-1) \\ \hline y(-n) & y(-n+1) & \dots & y(-n+T-1) \\ y(-n+1) & y(-n+2) & \dots & y(T-1) \\ \vdots & \vdots & \ddots & \vdots \\ y(-1) & u(0) & \dots & y(T-2) \\ u(-n) & u(-n+1) & \dots & u(-n+T-1) \\ u(-n+1) & u(-n+2) & \dots & u(T-1) \\ \vdots & \vdots & \ddots & \vdots \\ u(-1) & u(0) & \dots & u(T-2) \end{array}\right]$$

coincides with the matrix of <u>input-state</u> data of system $(\mathcal{A}, \mathcal{B})$

$$\left[\begin{array}{c} U_0 \\ \hline \Phi_0 \end{array}\right] := \left[\begin{array}{c|c|c|c} u(0) & u(1) & \dots & u(T-1) \\ \hline \phi(0) & \phi(1) & \dots & \phi(T-1) \end{array}\right]$$

since $\phi(k) = \begin{bmatrix} y(k-n) & \dots & y(k-1) & u(k-n) & \dots & u(k-1) \end{bmatrix}^\top$

- As $(\mathcal{A}, \mathcal{B})$ is reachable and $u_{[0,T-1]}$ PE of order $2n+1$, $\begin{bmatrix} U_0 \\ \Phi_0 \end{bmatrix}$ has full row rank.

Output feedback design

- Main idea Design a stabilizing "state" feedback controller
$$u(k) = \mathcal{K}\phi(k)$$
for the system
$$\phi(k+1) = \mathcal{A}\phi(k) + \mathcal{B}u(k)$$
- For each pair of matrices $\mathcal{K}$ and $\mathcal{G}$ that satisfy $\begin{bmatrix} \mathcal{K} \\ I_{2n} \end{bmatrix} = \begin{bmatrix} U_0 \\ \Phi_0 \end{bmatrix} \mathcal{G}$, the matrix $\mathcal{A} + \mathcal{B}\mathcal{K}$ results in the data-dependent representation
$$\mathcal{A} + \mathcal{B}\mathcal{K} = \Phi_1 \mathcal{G} \text{ with } \begin{bmatrix} \mathcal{K} \\ I_{2n} \end{bmatrix} = \begin{bmatrix} U_0 \\ \Phi_0 \end{bmatrix} \mathcal{G}$$
and
$$\Phi_1 = \begin{bmatrix} \ \phi(1) \mid \phi(2) \mid \ldots \mid \phi(T) \ \end{bmatrix}$$
- Using this representation and Lyapunov inequality, we conclude that a stabilizing $\mathcal{K}$ is computed as
$$\mathcal{K} = U_0 \mathcal{Y}^{-1} \mathcal{P}$$
where $\mathcal{P}, \mathcal{Y}$ are matrices that satisfy
$$\begin{bmatrix} \mathcal{P} & \Phi_1 \mathcal{Y} \\ \mathcal{Y}^\top \Phi_1^\top & \mathcal{P} \end{bmatrix} \succ 0 \quad \Phi_0 \mathcal{Y} = \mathcal{P}$$

State space representation of the controller

The controller

$$u(k) = \mathcal{K}\phi(k) = \begin{bmatrix} d_1 & \dots & d_n & -c_1 & \dots & -c_n \end{bmatrix} \begin{bmatrix} y(k-n) \\ \vdots \\ y(k-1) \\ u(k-n) \\ \vdots \\ u(k-1) \end{bmatrix}$$

is a Deterministic Auto Regressive Moving Average relation

$$\begin{aligned} u(k) + c_n u(k-1) + \ldots + c_2 u(k-n+1) + c_1 u(k-n) \\ = d_n y(k-1) + \ldots + d_2 y(k-n+1) + d_1 y(k-n) \end{aligned}$$

for which a state space realization in observable canonical form is computable

State space representation of the controller

The controller state vector

$$z^c(k) = \begin{bmatrix} u(k) \\ z_2^c(k) \\ \vdots \\ z_n^c(k) \end{bmatrix} = \begin{bmatrix} -c_n u(k-1) + d_n y(k-1) + z_2^c(k-1) \\ -c_{n-1} u(k-1) + d_{n-1} y(k-1) + z_3^c(k-1) \\ \vdots \\ -c_1 u(k-1) + d_1 y(k-1) \end{bmatrix}$$

returns the observable canonical form

$$z^c(k+1) = \underbrace{\begin{bmatrix} -c_n & 1 & 0 & \dots & 0 \\ -c_{n-1} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -c_2 & 0 & 0 & \dots & 1 \\ -c_1 & 0 & 0 & \dots 0 & 0 \end{bmatrix}}_{F} z^c(k) + \underbrace{\begin{bmatrix} d_n \\ d_{n-1} \\ \dots \\ d_2 \\ d_1 \end{bmatrix}}_{G} y(k)$$

$$u(k) = \underbrace{\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}}_{H} z^c(k)$$

Output feedback stabilization – main result

Theorem Consider the decision variables $\mathcal{P} \in \mathbb{R}^{2n\times 2n}$, $\mathcal{Y} \in \mathbb{R}^{T\times 2n}$ and the following SDP

$$\begin{bmatrix} \mathcal{P} & \Phi_1\mathcal{Y} \\ \mathcal{Y}^\top\Phi_1^\top & \mathcal{P} \end{bmatrix} \succ 0 \quad \Phi_0\mathcal{Y} = \mathcal{P}$$

If it is feasible, then the dynamic controller

$$\begin{aligned} z^c(k+1) &= \begin{bmatrix} -c_n & 1 & 0 & \dots & 0 \\ -c_{n-1} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -c_2 & 0 & 0 & \dots & 1 \\ -c_1 & 0 & 0 & \dots & 0 \end{bmatrix} z^c(k) + \begin{bmatrix} d_n \\ d_{n-1} \\ \vdots \\ d_2 \\ d_1 \end{bmatrix} y(k) \\ u(k) &= \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} z^c(k) \end{aligned}$$

with coefficients given by

$$\mathcal{K} = \begin{bmatrix} d_1 & \dots & d_n & -c_1 & \dots & -c_n \end{bmatrix} := U_0\mathcal{Y}\mathcal{P}^{-1}$$

renders the closed-loop system $x(k+1) = Ax(k) + BHz^c(k)$, $z^c(k+1) = Fz^c(k) + GCy(k)$ Schur stable.

Exercise #3

Exercise #3

(a) Consider the system

$$\dot{x} = Ax + Bu, y = Cx$$

in the observable canonical form (see answer to Exercise #2(a)) with state variable z. Find the matrix $V_1 \in \mathbb{R}^{n\times 2n}$ such that

$$z(k) = V_1\phi(k+n) \quad k \geq 0$$

Similarly, find the matrix $V_2 \in \mathbb{R}^{n\times 2n}$ such that

$$z^c(k) = V_2\phi(k+n) \quad k \geq 0$$

with z^c the state of the controller in the observable canonical from. Show that $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ is nonsingular.

(b) Show that the dynamical matrix of the closed-loop system

$$\begin{bmatrix} x(k+1) \\ z^c(k+1) \end{bmatrix} = \begin{bmatrix} A & BH \\ GC & F \end{bmatrix} \begin{bmatrix} x(k) \\ z^c(k) \end{bmatrix}$$

is similar to $\mathcal{A} + \mathcal{B}\mathcal{K}$, i.e. there exists a nonsingular matrix $\mathcal{S}$ such that

$$\begin{bmatrix} A & BH \\ GC & F \end{bmatrix} = \mathcal{S}(\mathcal{A} + \mathcal{B}\mathcal{K})\mathcal{S}^{-1}$$

Exercise #3 – answers

Answers

(a) We determine the matrix V_2. The same steps apply to the calculation of V_1. Consider the observable canonical form of the controller given before

$$
\begin{aligned}
z^c(k+1) &= \begin{bmatrix} -c_n & 1 & 0 & \dots & 0 \\ -c_{n-1} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -c_2 & 0 & 0 & \dots & 1 \\ -c_1 & 0 & 0 & \dots & 0 \end{bmatrix} z^c(k) + \begin{bmatrix} d_n \\ d_{n-1} \\ \vdots \\ d_2 \\ d_1 \end{bmatrix} y(k) \\
u(k) &= \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} z^c(k)
\end{aligned}
$$

We consider the expressions of the output u at time steps $k, k+1, \dots, k+n-1$:

$$
\begin{aligned}
u(k) &= z_1^c(k) \\
u(k+1) &= -c_n u(k) + z_2^c(k) + d_n y(k) \\
u(k+2) &= -c_n u(k+1) - c_{n-1} u(k) + z_3^c(k) + d_{n-1} y(k) + d_n y(k+1) \\
&\vdots \\
u(k+n-1) &= -c_n u(k+n-2) \dots - c_2 u(k) + z_n^c(k) + d_2 y(k) \dots \\
&\qquad + d_n y(k+n-2)
\end{aligned}
$$

Exercise #3 – answers

Solving for $z_c(k)$ one obtains

$$
\begin{aligned}
z_1^c(k) = &\quad u(k)\\
z_2^c(k) = &\quad u(k+1) + c_n u(k) - d_n y(k)\\
z_3^c(k) = &\quad u(k+2) + c_n u(k+1) + c_{n-1} u(k) - d_{n-1} y(k) - d_n y(k+1)\\
&\vdots\\
z_n^c(k) = &\quad u(k+n-1) + c_n u(k+n-2) \ldots + c_2 u(k) - d_2 y(k) \ldots - d_n y(k+n-2)
\end{aligned}
$$

Hence

$$
z^c(k) = \underbrace{\begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 & \ldots & 0\\ -d_n & 0 & 0 & \ldots & 0 & c_n & 1 & 0 & \ldots & 0\\ -d_{n-1} & -d_n & 0 & \ldots & 0 & c_{n-1} & c_n & 1 & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ -d_2 & -d_3 & -d_4 & \ldots & 0 & c_2 & c_3 & c_4 & \ldots & 1 \end{bmatrix}}_{V_2} \underbrace{\begin{bmatrix} y(k)\\ \vdots\\ y(k+n-1)\\ u(k)\\ \vdots\\ u(k+n-1) \end{bmatrix}}_{\phi(k+n)}
$$

Exercise #3 – answers

One similarly obtains

$$z(k) = \underbrace{\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ a_n & 1 & 0 & \dots & 0 & -b_n & 0 & 0 & \dots & 0 \\ a_{n-1} & a_n & 1 & \dots & 0 & -b_{n-1} & -b_n & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \dots & 1 & -b_2 & -b_3 & -b_4 & \dots & 0 \end{bmatrix}}_{V_1} \underbrace{\begin{bmatrix} y(k) \\ \vdots \\ y(k+n-1) \\ u(k) \\ \vdots \\ u(k+n-1) \end{bmatrix}}_{\phi(k+n)}$$

Hence

$$V = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ a_n & 1 & 0 & \dots & 0 & -b_n & 0 & 0 & \dots & 0 \\ a_{n-1} & a_n & 1 & \dots & 0 & -b_{n-1} & -b_n & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \dots & 1 & -b_2 & -b_3 & -b_4 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ -d_n & 0 & 0 & \dots & 0 & c_n & 1 & 0 & \dots & 0 \\ -d_{n-1} & -d_n & 0 & \dots & 0 & c_{n-1} & c_n & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -d_2 & -d_3 & -d_4 & \dots & 0 & c_2 & c_3 & c_4 & \dots & 1 \end{bmatrix}$$

Exercise #3 – answers

To show that V is nonsingular, take the Laplace expansion along the first row. Then

$$\det(V) = 1 \cdot \det(M_{11})$$

where M_{11} is the matrix obtained by removing the first column and the first row from V. Row n of M_{11}, which is row $n+1$ of V with the first element removed, is again a row with only one element different from zero and equal to 1. Taking the Laplace expansion along this row, we obtain again that

$$\det(V) = 1 \cdot 1 \cdot \det(M_{1\,n+1,1\,n+1})$$

where $M_{1\,n+1,1\,n+1}$ is the matrix obtained by removing rows and columns $1, n+1$ from V. $M_{1\,n+1,1\,n+1}$ has the first row with only one element different from zero and equal to 1, namely the element $(1,1)$ (the element $(2,2)$ of V), and this allows one to show that

$$\det(V) = 1 \cdot 1 \cdot 1 \cdot \det(M_{1\,2\,n+1,1\,2\,n+1})$$

where $M_{1\,2\,n+1,1\,2\,n+1}$ is the matrix obtained by removing rows and columns $1, 2, n+1$ from V. Iterating these arguments, one arrives at the result that $\det(V) = 1$.

Exercise #3 – answers

(b) Consider the closed-loop system

$$\begin{bmatrix} x(k+1) \\ z^c(k+1) \end{bmatrix} = \begin{bmatrix} A & BH \\ GC & F \end{bmatrix} \begin{bmatrix} x(k) \\ z^c(k) \end{bmatrix}$$

Show that its dynamic matrix is similar to $\mathcal{A} + \mathcal{B}\mathcal{K}$.
The state x of the process is related to vector z used for the observable canonical form by a nonsingular matrix S, i.e. $x = Sz$. Hence

$$\begin{bmatrix} x(k) \\ z^c(k) \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} z(k) \\ z^c(k) \end{bmatrix} = \underbrace{\begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} V}_{\mathcal{S}} \phi(k+n)$$

Then

$$\underline{\mathcal{S}^{-1} \begin{bmatrix} A & BH \\ GC & F \end{bmatrix} \begin{bmatrix} x(k) \\ z^c(k) \end{bmatrix}} = \mathcal{S}^{-1} \begin{bmatrix} x(k+1) \\ z^c(k+1) \end{bmatrix} = \phi(k+n+1) = (\mathcal{A}+\mathcal{B}\mathcal{K})\phi(k+n) = \underline{(\mathcal{A}+\mathcal{B}\mathcal{K})\mathcal{S}^{-1} \begin{bmatrix} x(k) \\ z^c(k) \end{bmatrix}}$$

Consider the underlined relation at $k = 0$. By the arbitrariness of $\left[\begin{smallmatrix} x(0) \\ z^c(0) \end{smallmatrix}\right]$, the relation above shows

$$\begin{bmatrix} A & BH \\ GC & F \end{bmatrix} = \mathcal{S}(\mathcal{A}+\mathcal{B}\mathcal{K})\mathcal{S}^{-1}$$

Output feedback stabilization of a mechanical system

Discretized version using a sampling time of 1sec of two carts mechanically coupled by a spring with uncertain stiffness. The output is the position of one of the carts and the input is a force applied to the other cart.

Here $n = 4$ and the (unknown) observable canonical form is specified by the coefficients

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} = \begin{bmatrix} 1 & -2.311 & 2.623 & -2.311 \end{bmatrix}$$
$$\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix} = \begin{bmatrix} 0.039 & 0.383 & 0.383 & 0.039 \end{bmatrix}$$

Data are generated from random initial conditions and by applying a random input sequence of length $T = 9$ and used to solve

$$\begin{bmatrix} \mathcal{P} & \Phi_1 \mathcal{Y} \\ \mathcal{Y}^\top \Phi_1^\top & \mathcal{P} \end{bmatrix} \succ 0 \quad \Phi_0 \mathcal{Y} = \mathcal{P} \text{ with } \Phi_0 = \left[\begin{array}{c|c|c|c} y(-4) & y(-3) & \dots & y(4) \\ y(-3) & y(-2) & \dots & y(5) \\ y(-2) & y(-2) & \dots & y(6) \\ y(-1) & y(0) & \dots & y(7) \\ u(-4) & u(-3) & \dots & u(4) \\ u(-3) & u(-2) & \dots & u(5) \\ u(-2) & u(-1) & \dots & u(6) \\ u(-1) & u(0) & \dots & u(7) \end{array}\right], \Phi_1 = \ldots$$

Output feedback stabilization of a mechanical system

Using CVX, we obtain from $\mathcal{K} = U_0\mathcal{Y}\mathcal{P}^{-1}$ with

$$U_0 = \begin{bmatrix} u(0) & u(1) & \dots & u(8) \end{bmatrix}$$

the "controller gain"

$$\mathcal{K} = \begin{bmatrix} 1.1837 & -1.5214 & 1.3408 & -1.4770 \\ 0.0005 & -0.5035 & -0.9589 & -0.9620 \end{bmatrix}$$

which stabilizes the closed-loop dynamics.

A minimal state-space representation (F, G, H) of this controller in the observable canonical form is given by

$$\left[\begin{array}{c|c} F & G \\ \hline H & 0 \end{array}\right] = \left[\begin{array}{c|c} \begin{bmatrix} -0.9620 & 1 & 0 & 0 \\ -0.9589 & 0 & 1 & 0 \\ -0.5035 & 0 & 0 & 1 \\ 0.0005 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} -1.4770 \\ 1.3408 \\ -1.5214 \\ 1.1837 \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} & 0 \end{array}\right]$$

A few remarks

- Under the assumption that $\begin{bmatrix} U_0 \\ \Phi_0 \end{bmatrix}$ has full row rank, any stabilizing output feedback controller can be given the form above with coefficients $\mathcal{K} = U_0 \mathcal{Y} \mathcal{P}^{-1}$
- Under suitable assumptions on the observability index, a similar design can be carried out in the case of Multiple Input Multiple Output systems
- As the output feedback stabilization problem is reduced to a "state" feedback stabilization problem, if we tackle the latter in the presence of noisy data, then we obtain a method to deal with noisy data for the former problem as well
- The design requires the knowledge of the state space dimension n. This is either available from physical principles or can be obtained from techniques processing the input-output data, as in e.g. subspace identification, without requiring the whole procedure to identify the system's model

Summary Lecture 1

Lecture 1

- Data-driven stabilization of linear systems via full state static feedback
- Data-driven LQR
- Data-driven stabilization of linear systems via partial state dynamic feedback

- Lectures 2 and 3 discuss how these results can be extended in the presence of process disturbances and noisy measurements
- Lecture 4 discusses extensions to nonlinear systems
- The results of Lecture 1 are taken from

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Formulas for Data-Driven Control: Stabilization, Optimality, and Robustness

Claudio De Persis and Pietro Tesi

A bridge towards Lecture 2

- The derivations in Lecture 1 were based on the data-dependent closed-loop system representation

$$x(k+1) = X_1 G_K x(k) \quad \text{with} \quad \begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G_K$$

- Suppose now that the system's dynamics is affected by disturbances

$$x(k+1) = Ax(k) + Bu(k) + d(k)$$

  How does the system's representation change?
  <u>Spoiler</u> The presence of noise leads to a perturbed data-dependent representation

$$x(k+1) = (X_1 - D_0) G_K x(k) \quad \text{with} \quad D_0 = \begin{bmatrix} d(0) & \dots d(T-1) \end{bmatrix}$$

- How would you design a controller for the system above if D_0 is unknown? Which new assumptions would you introduce?

Data driven control

linear and nonlinear systems

Lecture 2

C. De Persis°, P. Tesi◇

° Institute of Engineering and Technology
University of Groningen
◇ Department of Information Engineering
University of Firenze

with contributions by Andrea Bisoffi (Milan Polytechnic), Meichen Guo (TU Delft)

Data-driven control with noisy data

So far we have considered an ideal setting where the data are noiseless. In practice, we have to deal with noisy data, generated by disturbances and/or measurement noise:

$$\begin{aligned} x^+/\dot{x} &= Ax + Bu + d \qquad & x^+/\dot{x} &= f(x,u,d) \\ y &= Cx + Du + n \qquad & y &= h(x,u) + n \end{aligned}$$

Huge increase of complexity!

Problem overview

Suppose we have an LTI system

$$x(k+1) = A_\star x(k) + B_\star u(k) + d(k)$$

with $x, d \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. Suppose we perform an experiment of length T and collect the data matrices:

$$\begin{aligned} U_0 &:= \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix} \\ X_0 &:= \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix} \\ X_1 &:= \begin{bmatrix} x(1) & x(2) & \cdots & x(T) \end{bmatrix} \end{aligned}$$

Let D_0 be the unknown data matrix relative to d:

$$D_0 := \begin{bmatrix} d(0) & d(1) & \cdots & d(T-1) \end{bmatrix}$$

Objective Design K such that $A_\star + B_\star K$ is stable despite unknown D_0

Certainty-equivalence can fail

Both indirect (sys-ID-based) and direct methods fail to provide stability guarantees if we disregard noise.

Consider the method discussed in Lecture 1. The data-based relation for the system now reads:

$$\underbrace{\begin{bmatrix} x(1) & x(2) & \dots & x(T) \end{bmatrix}}_{X_1} = A_\star \underbrace{\begin{bmatrix} x(0) & x(1) & \dots & x(T-1) \end{bmatrix}}_{X_0} + B_\star \underbrace{\begin{bmatrix} u(0) & u(1) & \dots & u(T-1) \end{bmatrix}}_{U_0} + \underbrace{\begin{bmatrix} d(0) & d(1) & \dots & d(T-1) \end{bmatrix}}_{D_0}$$

In compact form:

$$X_1 = A_\star X_0 + B_\star U_0 + D_0$$

For any K:[1]

$$\begin{aligned} A_\star + B_\star K &= \begin{bmatrix} B_\star & A_\star \end{bmatrix} \begin{bmatrix} K \\ I_n \end{bmatrix} \\ &= \begin{bmatrix} B_\star & A_\star \end{bmatrix} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G_K \\ &= (X_1 - D_0) G_K \end{aligned}$$

where G_K satisfies $\left[\begin{smallmatrix} K \\ I_n \end{smallmatrix}\right] = \left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right] G_K$.

The Lyapunov condition thus reads

$$\underbrace{(X_1 - D_0) G_K P G_K^\top (X_1 - D_0)^\top - P}_{\mathcal{L}(G_K, P)} \prec 0$$

Simply solving $X_1 G_K P G_K^\top X_1^\top - P \prec 0$ does not ensure $\mathcal{L}(G_K, P) \prec 0$.

[1] assume $\left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right]$ full row rank

The very same issue arises with indirect methods.

Consider a least-squares approach: [2]

$$\underbrace{\begin{bmatrix}\hat{A} & \hat{B}\end{bmatrix}}_{\text{estimate}} = X_1 \begin{bmatrix}X_0 \\ U_0\end{bmatrix}^{\dagger}$$

$$\implies \underbrace{\begin{bmatrix}\hat{A} & \hat{B}\end{bmatrix} - \begin{bmatrix}A_\star & B_\star\end{bmatrix}}_{\text{estimate error}} = D_0 \begin{bmatrix}X_0 \\ U_0\end{bmatrix}^{\dagger}$$

Simply ensuring $\hat{A} + \hat{B}K$ stable does not ensure $A_\star + B_\star K$ stable.

We need to explicitly account for D_0

[2] M. Verhaegen, V. Verdult. Filtering and system identification: a least squares approach. Cambridge University Press, 2007.

Noise models and uncertainty

Since D_0 is unknown we can only assume that it belongs to a given class, the so-called noise model. There are many noise models, examples being Gaussian, unknown-but-bounded (UBB) ... Whatever the model, we now have uncertainty.

Example Suppose the true system is

$$x(k+1) = \underbrace{0.5}_{A_\star} x(k) + \underbrace{0.5}_{B_\star} u(k) + d(k)$$

We apply $u(0) = u(1) = 2$, and we measure $x(0) = 0$, $x(1) = 1$ and $x(2) = 2$. The state data have been generated by $d(0) = 0$, $d(1) = 0.5$. If we only know that $|\mathbf{d}| \leq \mathbf{1}$ then any (A, B) with

$$|1 - 2B| \leq 1, \; |2 - A - 2B| \leq 1$$

is also consistent with the data given our information on d.

Note $\begin{bmatrix} U_0 \\ X_0 \end{bmatrix}$ has full row rank.

Noise models and uncertainty

Since D_0 is unknown we can only assume that it belongs to a given class, the so-called noise model. There are many noise models, examples being Gaussian, unknown-but-bounded (UBB) ... Whatever the model, we now have uncertainty.

Example Suppose the true system is

$$x(k+1) = \underbrace{0.5}_{A_\star} x(k) + \underbrace{0.5}_{B_\star} u(k) + d(k)$$

We apply $u(0) = u(1) = 2$, and we measure $x(0) = 0$, $x(1) = 1$ and $x(2) = 2$. The state data have been generated by $d(0) = 0$, $d(1) = 0.5$. If we only know that $|\mathbf{d(0)}|^2 + |\mathbf{d(1)}|^2 \leq \mathbf{2}$ then any (A, B) with

$$(1 - 2B)^2 + (2 - A - 2B)^2 \leq 2,$$

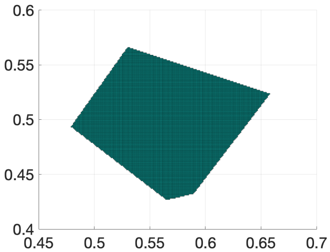
is also consistent with the data given our information on d.

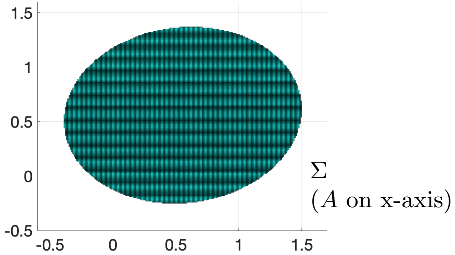
Note $\left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right]$ has full row rank.

The uncertainty takes the form of a consistency set

$$\Sigma := \{(A, B) \text{ consistent with the data given the noise model}\}$$

Example (Cont'd) Same system as before with $T = 100$ data points obtained with input and disturbances uniformly distributed within $[-1, 1]$. (Left) Set Σ assuming $|d| \leq 1$. (Right) Set Σ assuming $\sum_k d(k)^2 \leq 100$.





Robust data-driven control

In contrast with the noiseless case we must now design a controller that stabilizes a family of systems, namely we have a robust control problem.

In Lecture 2-3 we discuss how the approach introduced in Lecture 1 can be extended to handle noisy data by exploiting concepts and tools from robust control.

Same features as the baseline solution

- Conceptually simple
- Gives theoretical guarantees
- Easy to implement (SDP)
- Very flexible

Outline of Lecture 2 – 3

We will consider:

1. Input disturbances with energy-like models
2. Applications and extensions (brief overview)
3. Some recent results

restricting the analysis to full-state measurements.

Before we start

The results we are going to see also link to well-known concepts in sys-ID.
In fact, these results show that the difference between direct and indirect methods is quite mild.

Framework

Consider an LTI system

$$x(k+1) = A_\star x(k) + B_\star u(k) + d(k)$$

with $x, d \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. Suppose we perform an experiment of length T and collect the data matrices:

$$\begin{aligned}
U_0 &:= \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix} \\
X_0 &:= \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix} \\
X_1 &:= \begin{bmatrix} x(1) & x(2) & \cdots & x(T) \end{bmatrix}
\end{aligned}$$

Let $D_0 \in \mathbb{R}^{n \times T}$ be the <u>unknown</u> data matrix relative to d:

$$D_0 := \begin{bmatrix} d(0) & d(1) & \cdots & d(T-1) \end{bmatrix}$$

Objective Design K such that $A_\star + B_\star K$ is stable despite unknown D_0

Disturbance model

We consider a model that constrains the possible disturbance patterns in terms of an energy bound.

Disturbance model

$$\begin{aligned}\mathcal{D}_e &:= \left\{D \in \mathbb{R}^{n\times T} : DD^\top \preceq R_D R_D^\top \text{ for known } R_D\right\}\\ &= \left\{D \in \mathbb{R}^{n\times T} : \sum_{i=1}^{T} d_i d_i^\top \preceq R_D R_D^\top \text{ for known } R_D\right\}\end{aligned}$$

where d_i is the i-th column of D.

Note Bounded and convex set.

Energy bound If the disturbance signal $\{d_i\}_{i=0}^{T-1}$ has energy $\sum_{i=0}^{T-1} d_i^\top d_i \leq \gamma^2$, then $D \in \mathcal{D}_e$ with $R_D = \gamma\sqrt{T} I_n$.

Disturbance model

$$\mathcal{D}_e := \left\{ D \in \mathbb{R}^{n\times T} : DD^\top \preceq R_D R_D^\top \text{ for known } R_D \right\}$$
$$= \left\{ D \in \mathbb{R}^{n\times T} : \sum_{i=1}^{T} d_i d_i^\top \preceq R_D R_D^\top \text{ for known } R_D \right\}$$

where d_i is the i-th column of D.

Covers important cases:

- For $n = m$, the choice $R_D = \gamma U_0$ with $\gamma > 0$ gives an upper bound on the admissible input-disturbance SNR.
- Assuming $d \sim \mathcal{N}(0, \sigma^2 I_n)$, a "natural" choice is $R_D = \gamma\sqrt{T}\sigma I_n$, $\gamma > 1$.
- $R_D = \gamma\sqrt{T} I_n$ covers $\|d\| \leq \gamma$.

Consistency set

The relation for the true system reads:

$$\underbrace{\begin{bmatrix} x(1) & x(2) & \dots & x(T) \end{bmatrix}}_{X_1} =$$

$$A_\star \underbrace{\begin{bmatrix} x(0) & x(1) & \dots & x(T-1) \end{bmatrix}}_{X_0} + B_\star \underbrace{\begin{bmatrix} u(0) & u(1) & \dots & u(T-1) \end{bmatrix}}_{U_0}$$

$$+ \underbrace{\begin{bmatrix} d(0) & d(1) & \dots & d(T-1) \end{bmatrix}}_{D_0}$$

In compact form:

$$X_1 \;=\; A_\star X_0 + B_\star U_0 + D_0$$

Any other (A, B) that satisfies $X_1 = AX_0 + BU_0 + D$ for some $D \in \mathcal{D}_e$ is consistent with the dataset (U_0, X_0, X_1) and the noise model

Consistency set [a]

$$\begin{aligned}\Sigma_e &:= \{(A,B) : X_1 = AX_0 + BU_0 + D \text{ for some } D \in \mathcal{D}_e\} \\ &= \left\{(A,B) : X_1 = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} + D \text{ for some } D \in \mathcal{D}_e \right\}\end{aligned}$$

Note Convex set (as $\mathcal{D}_e$ is).

[a] Also termed 'Feasible Systems Set' in sys-ID

We will consider:

Assumpt. (Noise model correctness) $D_0 \in \mathcal{D}_e$
$(\Longleftrightarrow (A_\star, B_\star) \in \Sigma_e)$

Assumpt. (Quality of data) $W_0 := \left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right]$ has full row rank.

Note Implies Σ_e bounded.

Remark

Assuming $\left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right]$ full row rank is not strictly needed, but if the consistency set is unbounded the design problem is unlikely feasible. The assumption is also not restrictive. Note:

$$x^+ \;=\; A_\star x + B_\star u + d \;=\; A_\star x + \begin{bmatrix} B_\star & I_n \end{bmatrix} \begin{bmatrix} u \\ d \end{bmatrix}$$

If (u, d) is PE of order $n+1$ and $(A_\star, B_\star)$ is reachable, then

$$\operatorname{rank} \begin{bmatrix} U_0 \\ D_0 \\ X_0 \end{bmatrix} = 2n + m$$

ensuring $\left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right]$ full row rank.

Problem formulation

Problem I (Quadratic stabilization)

$$\begin{aligned} \text{find} \quad & K, P \succ 0 \\ \text{such that} \quad & \underbrace{(A+BK)P(A+BK)^\top - P}_{\mathcal{L}(K,P)} \prec 0 \quad \forall (A,B) \in \Sigma_e \end{aligned}$$

The difficult part is that we must ensure $\mathcal{L}(K,P) \prec 0$ for infinitely many systems, but that's exactly the realm of robust control!

Main tool: Petersen's lemma

Young's inequality (auxiliary)

or Generalized Square Inequality.

Lemma Let $M \in \mathbb{R}^{n \times p}$, $N \in \mathbb{R}^{q \times n}$ be given matrices. Consider the set $\mathcal{S} := \{S \in \mathbb{R}^{q \times p} : SS^\top \preceq R_S R_S^\top \text{ for given } R_s\}$. Then, for arbitrary $\varepsilon > 0$ it holds that

$$MS^\top N + N^\top S M^\top \preceq \varepsilon M M^\top + \varepsilon^{-1} N^\top R_S R_S^\top N \quad \forall S \in \mathcal{S}$$

Proof. A completion of squares

$$\left(\sqrt{\varepsilon} M - \sqrt{\varepsilon^{-1}} N^\top S\right)\left(\sqrt{\varepsilon} M - \sqrt{\varepsilon^{-1}} N^\top S\right)^\top \succeq 0$$

gives the result. $\square$

Petersen's lemma

Type of variable elimination method.

Lemma[3] Let $G = G^\top \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{n\times p}$, $N \in \mathbb{R}^{q\times n}$ be given matrices. Let $\mathcal{S} := \{S \in \mathbb{R}^{q\times p} : SS^\top \preceq R_S R_S^\top\}$. Then,

$$G + MS^\top N + N^\top S M^\top \prec 0 \quad \forall S \in \mathcal{S}$$

if and only if there exists $\varepsilon > 0$ such that

$$G + \varepsilon M M^\top + \varepsilon^{-1} N^\top R_S R_S^\top N \prec 0$$

Proof (sketch). Sufficiency follows from Young's inequality. Necessity is the difficult part, we only give some intuitions.

[3] I. Petersen, C. Hollot. A Riccati equation approach to the stabilization of uncertain linear systems, Automatica, 1986

We want to show that

$$\begin{aligned} & G + MS^\top N + N^\top SM^\top \prec 0 \quad \forall S \in \mathcal{S} \\ \implies \quad & \exists \varepsilon > 0 : G + \varepsilon MM^\top + \varepsilon^{-1} N^\top R_S R_S^\top N \prec 0 \end{aligned}$$

Consider the **scalar** case:[4]

$$\max_{S \in \mathcal{S}} G + MS^\top N + N^\top SM^\top < 0 \quad \implies \quad G + 2|M||R_S||N| < 0$$

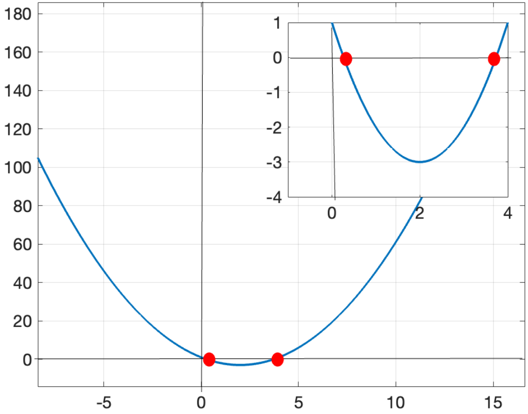
We also have $G - 2|M||R_S||N| < 0$, so

$$G^2 - 4M^2 R_S^2 N^2 > 0$$

This can be viewed as the discriminant of the second-order polynomial $M^2\lambda^2 + G\lambda + R_S^2 N^2$, which has two positive roots $\lambda_\pm$, with $\lambda_+ > \lambda_-$ (note that $M^2 > 0$, $G < 0$, $R_S^2 N^2 > 0$).

[4] assume $M, N, R_S \neq 0$

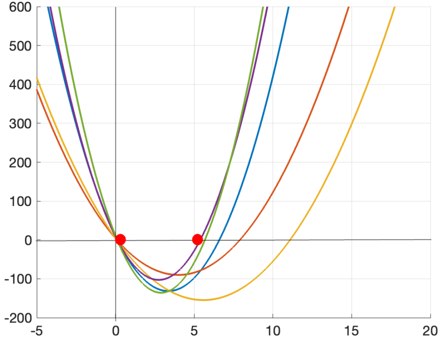
The polynomial $M^2\lambda^2 + G\lambda + R_S^2 N^2$ has two positive roots $\lambda_+ > \lambda_-$, so there exists a value $\varepsilon \in (\lambda_-, \lambda_+)$ such that $M^2\varepsilon^2 + G\varepsilon + R_S^2 N^2 < 0$. This is equivalent to $G + M^2\varepsilon + \varepsilon^{-1} R_S^2 N^2 < 0$.



In the **matrix** case, instead of $G^2 - 4M^2R_S^2N^2 > 0$ we have

$$\underbrace{(x^\top Gx)^2}_{b(x)^2} - 4\underbrace{(x^\top MM^\top x)}_{a(x)}\underbrace{(x^\top N^\top R_S R_S^\top Nx)}_{c(x)} > 0 \quad \forall x \neq 0$$

For each x, $p(x) := a(x)\lambda^2 + b(x)\lambda + c(x)$ has two roots $\lambda_+(x) > \lambda_-(x)$ and we can show that $\max_x \lambda_-(x) < \min_x \lambda_+(x)$. $\square$



Nonstrict Petersen's lemma

A version of Petersen's lemma with nonstrict inequalities also holds.

Lemma[5] Let $G = G^\top \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{n\times p}$, $N \in \mathbb{R}^{q\times n}$ be given matrices. Let $\mathcal{S} := \{S \in \mathbb{R}^{q\times p} : SS^\top \preceq R_S R_S^\top\}$. Suppose additionally that $M \neq 0$, $N \neq 0$ and $R_S R_S^\top \succ 0$. Then,

$$G + MS^\top N + N^\top S M^\top \prec 0 \quad \forall S \in \mathcal{S}$$

if and only if there exists $\varepsilon > 0$ such that

$$G + \varepsilon M M^\top + \varepsilon^{-1} N^\top R_S R_S^\top N \preceq 0$$

Proof (sketch). Sufficiency follows from Young's inequality. Necessity is the difficult part, we only give some intuitions.

[5] A. Bisoffi, C. De Persis P. Tesi. Data-driven control via Petersen's lemma, Automatica, 2022

Robust data-driven control design

Recall:

Framework

- Dynamics: $x^+ = A_\star x + B_\star u + d$
- Dataset: U_0, X_0, X_1
- Disturbance model: $\mathcal{D}_e = \left\{D : DD^\top \preceq R_D R_D^\top \text{ for known } R_D\right\}$
- Consistency set: $\Sigma_e = \{(A, B) : X_1 = AX_0 + BU_0 + D,\ D \in \mathcal{D}_e\}$

Problem I (Quadratic stabilization)

$$\begin{aligned} \text{find} \quad & K, P \succ 0 \\ \text{such that} \quad & (A + BK)P(A + BK)^\top - P \prec 0 \quad \forall (A, B) \in \Sigma_e \end{aligned}$$

Solution based on G_K-representation

The approach that we adopted for noise-free data (say G_K-representation) provides a solution also for the noisy case.

Recall $X_1 = A_\star X_0 + B_\star U_0 + D_0$. For any K:[6]

$$
\begin{aligned}
A_\star + B_\star K &= \begin{bmatrix} B_\star & A_\star \end{bmatrix} \begin{bmatrix} K \\ I_n \end{bmatrix} \\
&= \begin{bmatrix} B_\star & A_\star \end{bmatrix} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G_K \\
&= (X_1 - D_0) G_K
\end{aligned}
$$

where G_K satisfies

$$
\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G_K
$$

[6] we are assuming $\left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right]$ full row rank

This representation also holds for all (A, B) in the consistency set

$$\Sigma_e = \{(A, B) : X_1 = AX_0 + BU_0 + D,\ D \in \mathcal{D}_e\}$$

namely:

$$\begin{aligned} (A, B) \in \Sigma_e \quad \Longrightarrow \quad A + BK &= \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K \\ I_n \end{bmatrix} \\ &= \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G_K \\ &= (X_1 - D)G_K, \ \ D \in \mathcal{D}_e \end{aligned}$$

where G_K satisfies $\left[\begin{smallmatrix} K \\ I_n \end{smallmatrix}\right] = \left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right] G_K$.

Instead of

Problem I (Quadratic stabilization)

$$\begin{aligned} \text{find} \quad & K, P \succ 0 \\ \text{such that} \quad & (A + BK)P(A + BK)^\top - P \prec 0 \quad \forall (A, B) \in \Sigma_e \end{aligned}$$

we consider

Problem II (Quadratic stabilization via G_K-representation)

$$\begin{aligned} \text{find} \quad & G_K, P \succ 0 \\ \text{such that} \quad & (X_1 - D)G_K P G_K^\top (X_1 - D)^\top - P \prec 0 \quad \forall D \in \mathcal{D}_e \\ & X_0 G_K = I_n \end{aligned}$$

Note $K = U_0 G_K$ is set a posteriori.

It is simple to see that Problem II solves Problem I, and Problem II can be solved via Petersen's lemma.

Problem II $\Longrightarrow$ Problem I

Theorem Suppose $D_0 \in \mathcal{D}_e$. Then Problem II solves Problem I.

Proof. Suppose that Problem II has solution (G_K, P). Then $(X_1 - D)G_K$ is stable for all $D \in \mathcal{D}_e$. Let $K = U_0 G_K$ and recall:

$$(A, B) \in \Sigma_e \quad \Longrightarrow \quad A + BK = (X_1 - D)G_K,\ D \in \mathcal{D}_e$$

where G_K satisfies $\left[\begin{smallmatrix} K \\ I_n \end{smallmatrix}\right] = \left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right] G_K$.

Therefore $A + BK$ is stable for all $(A, B) \in \Sigma_e$. Since $D_0 \in \mathcal{D}_e$ then $A_\star + B_\star K$ is stable.
$\square$

Note Problem II is a relaxation of Problem I. Solving Problem II indeed amounts to ensuring

$$(X_1 - D)G_K \text{ stable } \forall D \in \mathcal{D}_e$$

However, for a given $D \in \mathcal{D}_e$, a pair (A, B) satisfying

$$X_1 = AX_0 + BU_0 + D$$

might not exist [7], meaning that we take into account more pairs (A, B) than what is strictly necessary.

[7] Exercise Show that it exists when $\ker\begin{bmatrix} U_0 \\ X_0 \end{bmatrix} \subseteq \ker(X_1 - D)$.

Main result

Theorem Consider a system $x^+ = A_\star x + B_\star u + d$ with dataset U_0, X_0, X_1. Consider the disturbance model $\mathcal{D}_e := \left\{D : DD^\top \preceq R_D R_D^\top \text{ for known } R_D\right\}$. Suppose $D_0 \in \mathcal{D}_e$.If there exist $P \in \mathbb{R}^{n\times n}, Y \in \mathbb{R}^{T\times n}$ such that

$$X_0 Y = P, \quad \begin{bmatrix} P - R_D R_D^\top & -X_1 Y & 0 \\ -Y^\top X_1^\top & P & Y^\top \\ 0 & Y & I \end{bmatrix} \succ 0$$

then $K = U_0 \underbrace{Y P^{-1}}_{G_K}$ is stabilizing.

<u>Note</u> Same structure as in the noiseless case

<u>Note</u> We stabilize all the systems compatible with the data in a 'set-membership' sense although the approach is still 'direct' (no explicit sys-ID)

Proof. Problem II is equivalent to finding matrices $G_K, P \succ 0$ such that $(X_1 - D)G_K P G_K^\top (X_1 - D)^\top - P \prec 0$ for all $D \in \mathcal{D}_e$, and $X_0 G_K = I_n$.

By the change of variable $Y = G_K P$, Problem II is equivalent to finding matrices $G_K, P \succ 0$ such that $(X_1 - D)YP^{-1}Y^\top (X_1 - D)^\top - P \prec 0$ for all $D \in \mathcal{D}_e$, and $X_0 G_K P = P$, which is equivalent to

$$X_0 Y = P, \quad \begin{bmatrix} -P & (X_1 - D)Y \\ Y^\top (X_1 - D)^\top & -P \end{bmatrix} \prec 0 \quad \forall D \in \mathcal{D}_e$$

The LMI reads

$$\underbrace{\begin{bmatrix} -P & X_1 Y \\ Y^\top X_1^\top & -P \end{bmatrix}}_{G} + \underbrace{\begin{bmatrix} I \\ 0 \end{bmatrix}}_{N^\top} D \underbrace{\begin{bmatrix} 0 & -Y \end{bmatrix}}_{M^\top} + \underbrace{\begin{bmatrix} 0 \\ -Y^\top \end{bmatrix}}_{M} D^\top \underbrace{\begin{bmatrix} I & 0 \end{bmatrix}}_{N} \prec 0$$

$$\forall D \in \mathcal{D}_e$$

By Petersen's lemma, the LMI is equivalent to

$$\exists \varepsilon > 0 : \quad \underbrace{\begin{bmatrix} -P & X_1 Y \\ Y^\top X_1^\top & -P \end{bmatrix}}_{G} + \varepsilon \underbrace{\begin{bmatrix} 0 \\ Y^\top \end{bmatrix} \begin{bmatrix} 0 & Y \end{bmatrix}}_{MM^\top} + \varepsilon^{-1} \underbrace{\begin{bmatrix} I \\ 0 \end{bmatrix}}_{N^\top} R_D R_D^\top \underbrace{\begin{bmatrix} I & 0 \end{bmatrix}}_{N} \prec 0$$

Overall, we have

$$X_0 Y = P, \quad \begin{bmatrix} -P + \varepsilon^{-1} R_D R_D^\top & X_1 Y \\ Y^\top X_1^\top & -P \end{bmatrix} + \varepsilon \begin{bmatrix} 0 \\ Y^\top \end{bmatrix} \begin{bmatrix} 0 & Y \end{bmatrix} \prec 0$$

A Schur complement finally gives

$$X_0 Y = P, \quad \begin{bmatrix} P - \varepsilon^{-1} R_D R_D^\top & -X_1 Y & 0 \\ -Y^\top X_1^\top & P & Y^\top \\ 0 & Y & \varepsilon^{-1} I \end{bmatrix} \succ 0$$

The result now follows by multiplying the LMI by ε, and letting $P \leftarrow \varepsilon P$ and $Y \leftarrow \varepsilon Y$.
$\square$

Example

Consider a randomly generated system

$$A_\star = \begin{bmatrix} -0.3245 & -0.5548 & -0.2793 \\ 0.5906 & -0.4228 & 0.0892 \\ -0.3792 & -0.2863 & -0.0984 \end{bmatrix}, \quad B_\star = \begin{bmatrix} 0.5864 \\ -0.8519 \\ 0.8003 \end{bmatrix}$$

We take $T = 100$ samples generated with $|u| \leq 1$ and $\|d\| \leq 0.1$. Assume that $R_D = I_3$ (the disturbance model is correct). The problem is feasible and we get a stabilizing controller $K = \begin{bmatrix} 0.2868 & 0.1201 & -0.4681 \end{bmatrix}$.

```
cvx_begin sdp
    variable P(n,n) symmetric
    variable Y(T,n)
    X0*Y == P;
    [P - R_D R_D', -X1*Y, zeros(n,T);
    -Y'*X1', P, Y';
    zeros(T,n), Y, eye(T)] > 0
cvx_end
K = U0*Y/P;
```

Remarks

- The SDP involves the decision variable $Y \in \mathbb{R}^{T \times n}$, thus the computational complexity increases with T. However, there is no advantage in using large datasets unless the disturbance has nice features. The quality of the data counts more.
- The LMI can be *infeasible* if our guess on the disturbance is too conservative. We must have model correctness ($D_0 \in \mathcal{D}_e$) and sufficiently tight bounds.

Applications and extensions (brief overview)

Some interesting features of this method

This method is independent of the noise statistics, it can equally handle deterministic noise, nonlinearities as well as stochastic noise:

- Examples of nonlinear systems
- Gaussian disturbances

It can also handle more complex noise settings:

- Measurement noise

Nonlinearities - stabilization by the first approximation

Consider a smooth nonlinear system

$$x(k+1) = f(x(k), u(k))$$

and let $(\overline{x}, \overline{u})$ be a *known* equilibrium pair, that is such that $\overline{x} = f(\overline{x}, \overline{u})$. We want to find K that locally stabilizes the system around $(\overline{x}, \overline{u})$. We can rewrite the nonlinear system as

$$\delta x(k+1) = A_\star \delta x(k) + B_\star \delta u(k) + d(k)$$

where $\delta x := x - \overline{x}$, $\delta u := u - \overline{u}$, and where

$$A_\star := \left.\frac{\partial f}{\partial x}\right|_{(x,u)=(\overline{x},\overline{u})}, \quad B_\star := \left.\frac{\partial f}{\partial u}\right|_{(x,u)=(\overline{x},\overline{u})}, \quad d := R(\delta x, \delta u) \begin{bmatrix} \delta x \\ \delta u \end{bmatrix}$$

with $R(\delta x, \delta u) \longrightarrow 0$ as $(\delta x, \delta u) \longrightarrow 0$. Same SDP as before.

- In this case good quality of data means that the experiments are conducted in a neighborhood of $(\overline{x}, \overline{u})$, so that d is sufficiently small
- Prior knowledge on the vector field $f(x, u)$ helps give a sufficiently tight bound R_D on the resulting D_0

Example

Consider the Euler discretization of an inverted pendulum:

$$\begin{aligned}
x_1(k+1) &= \quad x_1(k) + \Delta x_2(k) \\
x_2(k+1) &= \quad \frac{\Delta g}{\ell} \sin x_1(k) + \left(1 - \frac{\Delta \mu}{m\ell^2}\right) x_2(k) + \frac{\Delta}{m\ell^2} u(k)
\end{aligned}$$

- x_1 = angular velocity
- x_2 = angular position
- u = applied torque
- m = mass to be balanced
- ℓ = distance from the base to the centre of mass
- μ = coefficient of rotational friction
- g = acceleration due to gravity
- Δ = sampling time

The system has an unstable equilibrium in $(\overline{x}, \overline{u}) = (0, 0)$ corresponding to the pendulum upright position and thus $\delta x = x$, $\delta u = u$. Assume $\mu = 0.01$, $m = 1$ and $\ell = 1$ unknown.

The residual (disturbance term) is

$$d(k) = \begin{bmatrix} 0 \\ \dfrac{\Delta g}{\ell}\left(\sin x_1(k) - x_1(k)\right) \end{bmatrix}$$

We run a 2seconds-long experiment collecting $\overline{T} = 200$ samples ($\Delta = 0.01$) and use the first $T = 100$ samples ($\approx \pm 7^\circ$ displacement).

Over the experimental data, $\max_k(\sin x_1(k) - x_1(k)) \approx 1e\text{-}04$. Accounting for $\Delta g/\ell$, we expect $\|d(k)\| \leq \gamma := 1e\text{-}04$ over the experimental data. Hence we expect $\|D_0\| \leq \gamma\sqrt{T} = 1e\text{-}03$. We set $R_D := 0.01 I_2$. The problem is feasible and we obtain

$$K = \begin{bmatrix} -244.9589 & -38.6757 \end{bmatrix}$$

which is indeed stabilizing as $\|D_0\| = 4.6069e\text{-}05$.

Note Genuine disturbances can be included

Note PE for nonlinear systems is possible [8]

[8] C. De Persis, P. Tesi. Designing experiments for data-driven control of nonlinear systems, 24th MTNS 2020

Bilinear systems

Consider a single-input nonlinear system

$$x(k+1) = A_\star x(k) + B_\star u(k) + \underbrace{D_\star x(k)u(k)}_{d(k)}$$

with unknown $A_\star, B_\star, D_\star$. We want to locally stabilize the system around the origin. Let

$$V_0 := \begin{bmatrix} x(0)u(0) & x(1)u(1) & \cdots & x(T-1)u(T-1) \end{bmatrix}$$

Closed-loop representation:

$$\begin{aligned} A_\star + B_\star K + D_\star x K &= \begin{bmatrix} B_\star & A_\star \end{bmatrix} \begin{bmatrix} K \\ I_n \end{bmatrix} + \begin{bmatrix} D_\star x & 0 \end{bmatrix} \begin{bmatrix} K \\ I_n \end{bmatrix} \\ &= \begin{bmatrix} B_\star & A_\star \end{bmatrix} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G_K + \begin{bmatrix} D_\star x & 0 \end{bmatrix} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G_K \\ &= (X_1 - D_\star V_0 + D_\star x U_0) G_K \end{aligned}$$

The closed-loop representation reads

$$A_\star + B_\star K + D_\star x K = (X_1 - D_\star V_0 + D_\star x U_0) G_K$$

We consider as a Lyapunov function $V(x) = x^\top Q x$, $Q \succ 0$, and we apply Petersen's lemma twice considering

$$\mathcal{D}_e := \left\{ D \in \mathbb{R}^{n \times n} : DD^\top \preceq \delta^2 I_n \text{ with } \delta > 0 \right\}$$
$$\mathcal{E}_Q := \left\{ x \in \mathbb{R}^n : x^\top Q x \leq 1 \right\}$$

where:

- $\mathcal{D}_e$ replaces the term $D_\star$. It requires information on the strength of the *nonlinear coupling* (a Lipschitz constant).
- $\mathcal{E}_Q$ replaces the term x. It defines the *basin of attraction*.

Theorem[a] Consider a system $x^+ = A_\star x + B_\star u + D_\star xu$ with dataset $U_0, X_0\ X_1, V_0$. Consider the disturbance model $\mathcal{D}_e = \{D : DD^\top \preceq \delta^2 I_n,\ \delta > 0\}$ and suppose that $D_\star \in \mathcal{D}_e$. If there exist $\epsilon_1, \epsilon_2 \in \mathbb{R}$, $P \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{T\times n}$ such that

$$P = X_0 Y, \quad \begin{bmatrix} P & 0 & -Y^\top U_0^\top & -Y^\top X_1^\top & \delta Y^\top V_0^\top \\ \star & \epsilon_1 P & 0 & 0 & -\delta\epsilon_1 P \\ \star & \star & \epsilon_1 I & 0 & 0 \\ \star & \star & \star & P - \epsilon_2 I & 0 \\ \star & \star & \star & \star & \epsilon_2 I \end{bmatrix} \succ 0$$

then $K = U_0 Y \underbrace{P^{-1}}_{Q}$ is stabilizing and its basin of attraction contains the set

$$\mathcal{E}_{P^{-1}} = \{x : x^\top P^{-1} x \leq 1\}$$

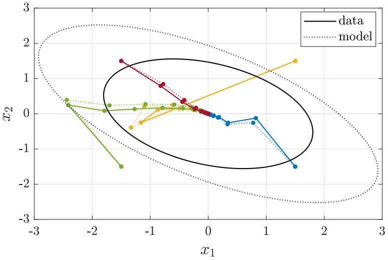
[a] A. Bisoffi, C. De Persis, P. Tesi. Data-based stabilization of unknown bilinear systems with guaranteed basin of attraction. Systems & Control Letters, 2020

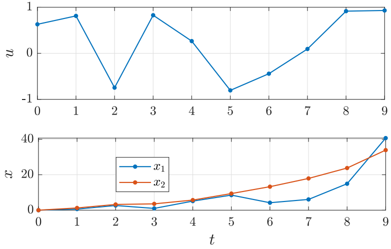
Example

Consider the system[9]

$$A_\star = \begin{bmatrix} 0.8 & 0.5 \\ 0.4 & 1.2 \end{bmatrix}, \; B_\star = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; D_\star = \begin{bmatrix} 0.45 & 0.45 \\ 0.3 & -0.3 \end{bmatrix}$$

We set $\delta := 0.763$ which over-approximates by 20% the actual $\|D_\star\| = 0.636$, and take $T = 10$ samples (unstable system). We solve the BMI using a line search on ϵ_1 obtaining $K = \begin{bmatrix} -0.3572 & -0.5738 \end{bmatrix}$.





[9] G. Bitsoris, N.Athanasopoulos. Constrained stabilization of bilinear discrete-time systems using polyhedral Lyapunov functions. 17th IFAC World Congress, 2008.

Gaussian noise

Assume that the disturbance vectors $d(k)$ are i.i.d. random vectors drawn from the normal distribution $\mathcal{N}(0, \sigma^2 I)$.

Theorem[a] For any $\mu > 0$ it holds that

$$D_0 D_0^\top \preceq \underbrace{\sigma^2 T \left(1 + \mu + \sqrt{\frac{n}{T}}\right)^2 I_n}_{R_D R_D^\top}$$

with probability at least $1 - e^{-T\mu^2/2}$.

[a] M. Wainwright. High-dimensional statistics: A non-asymptotic viewpoint. Cambridge University Press, 2019

We have an analogous result in probability ($D_0 \in \mathcal{D}_e$ with high probability)

There is a tradeoff on the choice of the length of the experiment: the higher T, the higher the probability that $D_0 D_0^\top \preceq R_D R_D^\top$, but the higher is the value of $\|R_D\| = \sigma\sqrt{T}(1 + \mu) + \sigma\sqrt{n}$. Things improve via averaging.

Example

Consider a linear system with

$$A_\star = \begin{bmatrix} -0.3245 & -0.5548 & -0.2793 \\ 0.5906 & -0.4228 & 0.0892 \\ -0.3792 & -0.2863 & -0.0984 \end{bmatrix}, \quad B_\star = \begin{bmatrix} 0.5864 \\ -0.8519 \\ 0.8003 \end{bmatrix}$$

We perform an experiment with input $|u| \leq 1$ and we collect $T = 100$ samples. We assume that $d \sim \mathcal{N}(0, 0.01 I_3)$ i.i.d.. For $\mu = 0.4$ we obtain $\|D_0\| \leq 1.5732$ with probability at least 0.999. We set $R_D := 1.5732 I_3$. The problem is feasible and we get

$$K = \begin{bmatrix} 0.4768 & -0.0018 & -0.0451 \end{bmatrix}$$

This controller ensures closed-loop stability with 99.9% probability. It is indeed stabilizing as $\|D_0\| = 0.9972$.

Averaging

We can filter out noise by averaging datasets from multiple experiments. [10]

Take N independent experiments each of length T with datasets $(U_0^{(n)}, X_0^{(n)}, X_1^{(n)}, D_0^{(n)})$, $n = 1, \ldots, N$. Denoting

$$\underline{S} = \frac{1}{N} \sum_{n=1}^{N} S^{(n)}$$

we obtain the relation

$$\underline{X}_1 = A_\star \underline{X}_0 + B_\star \underline{U}_0 + \underline{D}_0$$

The average signals still provide a valid trajectory of the system and the noise will now have a reduced variance.

[10] C De Persis, P Tesi. Low-complexity learning of linear quadratic regulators from noisy data. Automatica 2021

Easy consequence of the previous result:

Theorem[a] Consider N independent experiments each of length T and assume that the disturbance vectors $d(k)$ are i.i.d. random vectors drawn from the normal distribution $\mathcal{N}(0, \sigma^2 I)$. For any $\mu > 0$, the average matrix $\underline{D}_0$ satisfies

$$\underline{D}_0 \underline{D}_0^\top \preceq \frac{\sigma^2 T}{N} \left(1 + \mu + \sqrt{\frac{n}{T}}\right)^2 I_n$$

with probability at least $1 - e^{-T\mu^2/2}$.

(*Averaging reduces noise power by a factor of* N)

[a] M. Wainwright. High-dimensional statistics: A non-asymptotic viewpoint. Cambridge University Press, 2019. Theorem 6.1

Example (Cont'd)

Consider the same system as before

$$A_\star = \begin{bmatrix} -0.3245 & -0.5548 & -0.2793 \\ 0.5906 & -0.4228 & 0.0892 \\ -0.3792 & -0.2863 & -0.0984 \end{bmatrix}, \quad B_\star = \begin{bmatrix} 0.5864 \\ -0.8519 \\ 0.8003 \end{bmatrix}$$

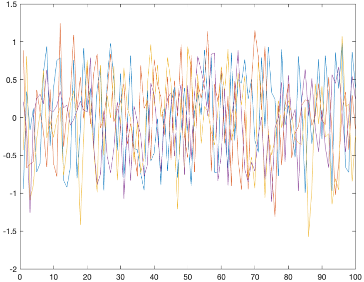
We take again $|u| \le 1$ and $T = 100$, but this time we assume $d \sim \mathcal{N}(0, 0.3 I_3)$ (instead of $d \sim \mathcal{N}(0, 0.01 I_3)$). We consider $N = 100$ experiments (same u). For $\mu = 0.4$ we have $\|\underline{D}_0\| \le 0.7425$ with probability at least 0.999. We set $R_D := 0.7425 I_3$ and obtain

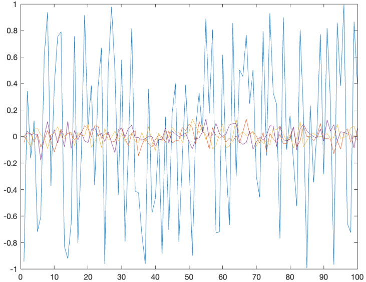
$$K = \begin{bmatrix} 0.4017 & 0.0422 & -0.1221 \end{bmatrix}$$

The controller ensures closed-loop stability with 99.9% probability. It is indeed stabilizing as $\|\underline{D}_0\| = 0.6163$.

(Left) Input and disturbance signals for one of the experiments

(Right) Input and disturbance signals after averaging $N = 100$ experiments





Measurement noise

Consider the following setting:

$$x^{+} = A_\star x + B_\star u, \quad y = x + n$$

The relation between data and dynamics now reads:

$$X_1 = A_\star X_0 + B_\star U_0, \quad Y_0 = X_0 + N_0, \quad Y_1 = X_1 + N_1$$

This gives the following identity relating the measured noisy data matrices Y_0, Y_1 (instead of X_0, X_1)

$$\begin{aligned} Y_1 &= X_1 + N_1 \\ &= A_\star X_0 + B_\star U_0 + N_1 \\ &= A_\star Y_0 + B_\star U_0 + \underbrace{N_1 - A_\star N_0}_{Q_0 = Q_0(N_0, N_1, A_\star)} \end{aligned}$$

Same structure as before although the problem itself is now much more complex (consistency set is non-convex).

Theorem [a] Consider $x^+ = A_\star x + B_\star u$, $y = x + n$, with dataset U_0, Y_0, Y_1 and consider the noise model $\mathcal{Q}_e := \left\{Q : QQ^\top \preceq R_Q R_Q^\top \text{ with known } R_Q\right\}$. Suppose $Q_0 \in \mathcal{Q}_e$. If there exist $P \in \mathbb{R}^{n\times n}, Z \in \mathbb{R}^{T\times n}$ such that

$$Y_0 Z = P, \quad \begin{bmatrix} P - R_Q R_Q^\top & -Y_1 Z & 0 \\ -Z^\top Y_1^\top & P & Z^\top \\ 0 & Z & I \end{bmatrix} \succ 0$$

then $K = U_0 \underbrace{Z P^{-1}}_{G_K}$ is stabilizing.

[a] C. De Persis, P. Tesi. Formulas for data-driven control: Stabilization, optimality, and robustness. TAC 2020

We have

$$Y_1 = A_\star Y_0 + B_\star U_0 + \underbrace{N_1 - A_\star N_0}_{Q_0}$$

and the model $\mathcal{Q}_e = \{Q : QQ^\top \preceq R_Q R_Q^\top\}$.

Alternative We can take the model $\mathcal{N}_e = \{N : NN^\top \preceq R_N R_N^\top\}$. If $N_0, N_1 \in \mathcal{N}_e$ and there exists $\epsilon \in (0, 0.5)$ such that

$$\begin{bmatrix} 0 \\ R_N \end{bmatrix} \begin{bmatrix} 0 \\ R_N \end{bmatrix}^\top \preceq \epsilon \begin{bmatrix} U_0 \\ Y_0 \end{bmatrix} \begin{bmatrix} U_0 \\ Y_0 \end{bmatrix}^\top, \quad R_N R_N^\top \preceq \epsilon Y_1 Y_1^\top$$

then $Q_0 \in \mathcal{Q}_e$ with $R_Q = \gamma Y_1$, where $\gamma = 9\epsilon/(1 - 2\epsilon)$.

This approach avoids to infer bounds on the norm of $A_\star$

Example

Consider the same system as before

$$A_\star = \begin{bmatrix} -0.3245 & -0.5548 & -0.2793 \\ 0.5906 & -0.4228 & 0.0892 \\ -0.3792 & -0.2863 & -0.0984 \end{bmatrix}, \quad B_\star = \begin{bmatrix} 0.5864 \\ -0.8519 \\ 0.8003 \end{bmatrix}$$

We run an experiment with $|u| \leq 1$ and we collect $T = 100$ samples. Suppose that $\|n\| \leq 0.01$ and that we select $R_N = 0.1 I_3$ (very large over-approximation). We find a feasible value $\epsilon = 0.0179$. Hence we set $\gamma := 0.1671$ and $R_Q := \gamma Y_1$, which corresponds to $\approx 16\%$ information loss relative to the observations.

The problem is feasible and we get

$$K = \begin{bmatrix} 0.5258 & -0.0192 & 0.0401 \end{bmatrix}$$

The controller ensures closed-loop stability as R_N is a correct guess.

Summary

- The same approach used for noise-free data extends to the noisy case and provides theoretical guarantees.
- As in the noiseless case, it only requires a finite number of data from a low sample-complexity experiment.
- It equally handles deterministic and stochastic noise.

References

Most of the material (noisy measurements, stabilization by the first approximation) is taken/adapted from:

C. De Persis, P. Tesi. Formulas for data-driven control: Stabilization, optimality, and robustness. IEEE TAC 2019

The use of the Petersen's lemma has been proposed for bilinear systems in:

A. Bisoffi, C. De Persis, P. Tesi. Data-based stabilization of unknown bilinear systems with guaranteed basin of attraction. Systems & Control Letters 2020

The case of Gaussian noise and the G_K-representation for optimal control (LQR) is treated in:

C. De Persis, P. Tesi. Low-complexity learning of linear quadratic regulators from noisy data. Automatica 2021

Some recent results

This approach provides sufficient conditions. In general, we do not expect tractable NS conditions. Tight conditions are nevertheless very important to tackle more challenging (nonlinear) settings.

Two additional findings for LTI systems with perturbed dynamics:

1. Tight conditions for energy models
2. Tractable conditions for point-wise bounds

We review these findings in Lecture 3

Other results

G_K-representations have been used for other classes of nonlinear systems

Nonlinear input affine systems, polynomial systems (lecture 4)

M. Guo, C. De Persis, P. Tesi. Learning control for polynomial systems using sum of square. IEEE Conference in Decision and Control 2020.
M. Guo, C. De Persis, P. Tesi. Data-driven stabilization of nonlinear polynomial systems with noisy data. IEEE Transactions on Automatic Control, 2022.

Lur'e systems

A. Luppi, C. De Persis, P. Tesi. On data-driven stabilization of systems with nonlinearities satisfying quadratic constraints. Systems & Control Letters, 2022.

Nonlinear systems with no specific functions or structure: approximate feedback linearization

C. De Persis, M. Rotulo, P. Tesi. Learning controllers from data via approximate nonlinearity cancellation. IEEE Transactions on Automatic Control 2023

Kernel-based methods

Z. Hu, C. De Persis, P. Tesi. Learning controllers from data via kernel-based interpolation. IEEE Conference in Decision and Control 2023.

As well as for other problems

Event-triggered control

C. De Persis, R. Postoyan, P. Tesi. Event-triggered control from data. IEEE Transactions on Automatic Control (provisionally accepted), arXiv:2208.11634

Data driven control

linear and nonlinear systems

Lecture 3

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Some recent results

This approach provides sufficient conditions. In general, we do not expect tractable NS conditions. Tight conditions are nevertheless very important to tackle more challenging (nonlinear) settings.

Two additional findings for LTI systems with perturbed dynamics:

1. Tight conditions for energy models
2. Tractable conditions for point-wise bounds

Tight conditions for energy models

Tight conditions are actually possible for linear systems in case of input disturbances and energy-like model.

- Approach based on Petersen's lemma.
- Also useful for nonlinear (polynomial) systems which will be discussed in the last part of the lectures.

Framework and problem recap

Consider the LTI system

$$x(k+1) = A_\star x(k) + B_\star u(k) + d(k)$$

where $x, d \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

Disturbance model and consistency set are given by:

Disturbance model

$$\mathcal{D}_e := \left\{ D \in \mathbb{R}^{n \times T} : DD^\top \preceq R_D R_D^\top \text{ for known } R_D \right\}$$

where T is the experiment length.

Consistency set

$$\Sigma_e := \{(A, B) : X_1 = AX_0 + BU_0 + D \text{ for some } D \in \mathcal{D}_e\}$$

Problem I (Quadratic stabilization)

$$\begin{aligned} \text{find} \quad & K, P \succ 0 \\ \text{such that} \quad & (A+BK)P(A+BK)^\top - P \prec 0 \quad \forall (A,B) \in \Sigma_e \end{aligned}$$

Problem II (Quadratic stabilization via G_K-representation)

$$\begin{aligned} \text{find} \quad & G_K, P \succ 0 \\ \text{such that} \quad & (X_1 - D)G_K P G_K^\top (X_1 - D)^\top - P \prec 0 \quad \forall D \in \mathcal{D}_e \\ & X_0 G_K = I_n \end{aligned}$$

We know that Problem II $\Longrightarrow$ Problem I but Problem II $\not\Longleftarrow$ Problem I. Instead of applying Petersen's lemma to D we might apply it to (A,B). We must better understand the structure of Σ_e.

An equivalent description of Σ_e

Lemma $\Sigma_e = \Lambda_e$, where

$$\Lambda_e := \left\{ (A,B) : \begin{bmatrix} I_n \\ B^\top \\ A^\top \end{bmatrix}^\top \begin{bmatrix} I_n & X_1 \\ 0 & -U_0 \\ 0 & -X_0 \end{bmatrix} \begin{bmatrix} -R_D R_D^\top & 0 \\ 0 & I_T \end{bmatrix} \begin{bmatrix} I_n & X_1 \\ 0 & -U_0 \\ 0 & -X_0 \end{bmatrix}^\top \begin{bmatrix} I_n \\ B^\top \\ A^\top \end{bmatrix} \preceq 0 \right\}$$

Proof. Assume that $(A,B) \in \Sigma_e = \{(A,B) : X_1 = AX_0 + BU_0 + D,\ D \in \mathcal{D}_e\}$. Then

$$\left\{ \begin{array}{l} D = X_1 - \left[\begin{smallmatrix} B & A \end{smallmatrix}\right] \left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right] \\ \begin{bmatrix} I_n \\ D^\top \end{bmatrix}^\top \begin{bmatrix} -R_D R_D^\top & 0 \\ 0 & I_T \end{bmatrix} \begin{bmatrix} I_n \\ D^\top \end{bmatrix} \preceq 0 \end{array} \right.$$

Replacing the expression of D in the identity in the inequality above gives $(A,B) \in \Lambda_e$.

Let $(A,B) \in \Lambda_e$, define $D := X_1 - AX_0 - BU_0$ and note that

$$\begin{bmatrix} I_n & X_1 \\ 0 & -U_0 \\ 0 & -X_0 \end{bmatrix}^\top \begin{bmatrix} I_n \\ B^\top \\ A^\top \end{bmatrix} = \begin{bmatrix} I_n \\ D^\top \end{bmatrix}$$

Hence, $D \in \mathcal{D}_e$ and this implies $(A,B) \in \Sigma_e$. $\square$

The constraint defining Σ_e is then

$$
\begin{aligned}
&\begin{bmatrix} I_n \\ B^\top \\ A^\top \end{bmatrix}^\top \begin{bmatrix} I_n & X_1 \\ 0 & -U_0 \\ 0 & -X_0 \end{bmatrix} \begin{bmatrix} -R_D R_D^\top & 0 \\ 0 & I_T \end{bmatrix} \begin{bmatrix} I_n & X_1 \\ 0 & -U_0 \\ 0 & -X_0 \end{bmatrix}^\top \begin{bmatrix} I_n \\ B^\top \\ A^\top \end{bmatrix} \preceq 0 \\
\iff \quad & \left[\begin{array}{c} I_n \\ \hline B^\top \\ A^\top \end{array}\right]^\top \left[\begin{array}{c|cc} X_1 X_1^\top - R_D R_D^\top & -X_1 U_0^\top & -X_1 X_0^\top \\ \hline -U_0 X_1^\top & U_0 U_0^\top & U_0 X_0^\top \\ -X_0 X_1^\top & X_0 U_0^\top & X_0 X_0^\top \end{array}\right] \left[\begin{array}{c} I_n \\ \hline B^\top \\ A^\top \end{array}\right] \preceq 0 \\
\iff \quad & \left[\begin{array}{c} I_n \\ \hline \Delta \end{array}\right]^\top \left[\begin{array}{c|c} \Xi & V^\top \\ \hline V & \Theta \end{array}\right] \left[\begin{array}{c} I_n \\ \hline \Delta \end{array}\right] \preceq 0 \\
\iff \quad & \Xi + \Delta^\top V + V^\top \Delta + \Delta^\top \Theta \Delta \preceq 0 \\
\iff \quad & \left(\Delta + \Theta^{-1} V\right)^\top \Theta \left(\Delta + \Theta^{-1} V\right) + \Xi - V^\top \Theta^{-1} V \preceq 0
\end{aligned}
$$

where the last expressions follows from $\left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right]$ full row rank ($\Theta = \left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right] \left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right]^\top$).

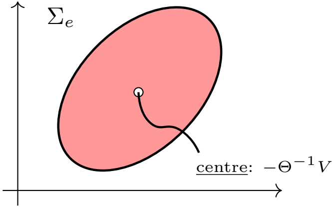
Ellipsoidal description of Σ_e

We have

$$\Sigma_e = \left\{\Delta \in \mathbb{R}^{(n+m)\times n} : \left(\Delta + \Theta^{-1}V\right)^\top \Theta \left(\Delta + \Theta^{-1}V\right) \preceq L\right\}$$

where:

- $\Delta := \begin{bmatrix} B & A \end{bmatrix}^\top$
- $\Theta := W_0 W_0^\top$
- $V := -W_0 X_1^\top$
- $L := V^\top \Theta^{-1} V - \Xi$
- $\Xi := X_1 X_1^\top - R_D R_D^\top$



We have an ‘ellipsoidal’ uncertainty: $\mathcal{E} = \{\delta : (\delta - \delta_c)^\top \Theta(\delta - \delta_c) \leq 1\}$. The centre is $-\Theta^{-1}V$ and the size depends on L.

Note $L \succeq 0$.
If $L = 0$, then Σ_e reduces to a singleton (noiseless case)
If $L \succ 0$, then $\Delta = -\Theta^{-1}V + \Theta^{-1/2}SL^{1/2}$, for any $S \in \mathbb{R}^{(n+m)\times n}$ such that $\|S\| \leq 1$, belongs to Σ_e and implies that Σ_e has nonempty interior (”in all directions”).

There is more than one way to represent an ellipsoid, and there exists an expression that is suited to apply Petersen's lemma.

Lemma It holds that

$$\Sigma_e = \Delta_e$$

where

$$\Delta_e := \left\{ \Delta \in \mathbb{R}^{(n+m)\times n} : \Delta = \underbrace{-\Theta^{-1}V + \Theta^{-1/2}SL^{1/2},\ \|S\| \leq 1}_{\text{explicit description}} \right\}$$

Proof (we only consider $L \succ 0$). We want to prove that

$$\Sigma_e = \left\{ \Delta : (\Delta + \Theta^{-1}V)^\top \Theta (\Delta + \Theta^{-1}V) \preceq L \right\}$$
$$= \left\{ \Delta : \Delta = -\Theta^{-1}V + \Theta^{-1/2} S L^{1/2},\ \|S\| \leq 1 \right\} =: \Delta_e$$

Assume $\Delta \in \Sigma_e$. Let

$$S := \Theta^{1/2}(\Delta + \Theta^{-1}V)L^{-1/2}$$

Clearly, $\Delta = -\Theta^{-1}V + \Theta^{-1/2}SL^{1/2}$. We just need to check $\|S\| \leq 1$:

$$S^\top S = L^{-1/2}(\Delta + \Theta^{-1}V)^\top \Theta (\Delta + \Theta^{-1}V)L^{-1/2} \preceq I_n$$

Hence, $\Delta \in \Delta_e$.

Assume $\Delta \in \Delta_e$. Then,

$$(\Delta + \Theta^{-1}V)^\top \Theta (\Delta + \Theta^{-1}V) = L^{1/2} S^\top S L^{1/2} \preceq L$$

Hence, $\Delta \in \Sigma_e$. $\square$

Main result

Theorem[a] Consider a system $x^+ = A_\star x + B_\star u + d$ with dataset U_0, X_0, X_1. Assume that $W_0 = \left[\begin{smallmatrix} U_0 \\ X_0 \end{smallmatrix}\right]$ has full row rank, and assume that $D_0 \in \mathcal{D}_e$ where $\mathcal{D}_e := \{D : DD^\top \preceq R_D R_D^\top \text{ for known } R_D\}$. Problem I is feasible if and only if there exist $P \in \mathbb{R}^{n\times n}, Y \in \mathbb{R}^{m\times n}$ such that

$$\begin{bmatrix} P + \Xi & 0 & V^\top \\ 0 & P & \begin{pmatrix} Y^\top & P \end{pmatrix} \\ V & \begin{pmatrix} Y \\ P \end{pmatrix} & \Theta \end{bmatrix} \succ 0$$

where $V = -W_0 X_1^\top$, $\Xi = X_1 X_1^\top - R_D R_D^\top$, $\Theta = W_0 W_0^\top$.

In this case, $K = Y P^{-1}$.

[a] A. Bisoffi, C. De Persis, P. Tesi. Data-driven control via Petersen's lemma. Automatica 2022

Proof (sketch). For any $(A, B) \in \Sigma_e$,

$$A + BK = \Delta \begin{bmatrix} K \\ I \end{bmatrix} = (-\Theta^{-1}V + \Theta^{-1/2}SL^{1/2})\underline{K} \text{ for some } S \text{ such that } \|S\| \leq 1$$

where $\underline{K} := \left[\begin{smallmatrix} K \\ I \end{smallmatrix}\right]$.

The Lyapunov stability condition reads

$$\underbrace{\Theta^{-1/2}SL^{1/2} - \Theta^{-1}V)^\top}_{[\,B \;\; A\,]}\underline{K}P\underline{K}^\top(\Theta^{-1/2}SL^{1/2} - \Theta^{-1}V) - P \prec 0$$

$$\text{for all } S \text{ with } \|S\| \leq 1$$

Let now $\underline{Y} = \underline{K}P = \left[\begin{smallmatrix} KP \\ P \end{smallmatrix}\right] =: \left[\begin{smallmatrix} Y \\ P \end{smallmatrix}\right]$, which implies $\underline{K}P\underline{K}^\top = \underline{Y}P^{-1}\underline{Y}^\top$. A Schur complement gives

$$\begin{bmatrix} -P & (\Theta^{-1}V - \Theta^{-1/2}SL^{1/2})^\top\underline{Y} \\ \underline{Y}^\top(\Theta^{-1}V - \Theta^{-1/2}SL^{1/2}) & -P \end{bmatrix}$$

$$= \begin{bmatrix} -P & V^\top\Theta^{-1}\underline{Y} \\ \underline{Y}^\top\Theta^{-1}V & -P \end{bmatrix} + \begin{bmatrix} L^{1/2} \\ 0 \end{bmatrix} {\color{red}S^\top} \begin{bmatrix} 0 & -\Theta^{-1/2}\underline{Y} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ -\underline{Y}^\top\Theta^{-1/2} \end{bmatrix} {\color{red}S} \begin{bmatrix} L^{1/2} & 0 \end{bmatrix} \prec 0$$

$$\text{for all } S \text{ with } \|S\| \leq 1$$

By Petersen's lemma, this condition is equivalent to the existence of a scalar $\varepsilon > 0$ such that

$$\begin{bmatrix} -P & V^\top \Theta^{-1} \underline{Y} \\ \underline{Y}^\top \Theta^{-1} V & -P \end{bmatrix} + \varepsilon^{-1} \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 \\ \underline{Y}^\top \Theta^{-1/2} \end{bmatrix} I_{n+m} \begin{bmatrix} 0 & \Theta^{-1/2} \underline{Y} \end{bmatrix} \prec 0$$

This inequality can be rewritten as

$$\begin{bmatrix} -P + \varepsilon^{-1} L & V^\top \Theta^{-1} \underline{Y} \\ \underline{Y}^\top \Theta^{-1} V & -P + \varepsilon \underline{Y}^\top \Theta^{-1} \underline{Y} \end{bmatrix} \prec 0 \text{ or } \begin{bmatrix} -P + L & V^\top \Theta^{-1} \underline{Y} \\ \underline{Y}^\top \Theta^{-1} V & -P + \underline{Y}^\top \Theta^{-1} \underline{Y} \end{bmatrix} \prec 0$$

the latter by scaling the decision variables $\varepsilon P \to P$, $\varepsilon \underline{Y} \to \underline{Y}$. Rewrite it as

$$\begin{bmatrix} -P + L - V^\top \Theta^{-1} V & 0 \\ \underline{Y}^\top \Theta^{-1} V & -P + \underline{Y}^\top \Theta^{-1} \underline{Y} \end{bmatrix} + \begin{bmatrix} -V^\top \\ -\underline{Y}^\top \end{bmatrix} \Theta^{-1} \begin{bmatrix} -V & -\underline{Y} \end{bmatrix} \prec 0$$

which shows the result after another Schur complement and bearing in mind that $L - V^\top \Theta^{-1} V = -\Xi$. $\square$

Example

Consider again the system

$$A_\star = \begin{bmatrix} -0.3245 & -0.5548 & -0.2793 \\ 0.5906 & -0.4228 & 0.0892 \\ -0.3792 & -0.2863 & -0.0984 \end{bmatrix}, \quad B_\star = \begin{bmatrix} 0.5864 \\ -0.8519 \\ 0.8003 \end{bmatrix}$$

We collect $T = 100$ samples with $|u| \leq 1$, $\|d\| \leq 0.1$. Suppose $R_D = \gamma I_3$. With the G_K-representation the problem is feasible up to $\gamma = 2.18$, while the Δ-representation works up to $\gamma = 2.37$.

<u>Note</u> The gap is not large. The reason is that Problem II is a relaxation of Problem I but we have NS conditions for Problem II.

Remarks

Some practical advantages:

- Tight conditions
- Reduced number of decision variables ($Y \in \mathbb{R}^{m \times n}$)
- Useful for linear-like (polynomial) systems

It also allows us to draw connections with least-squares methods.

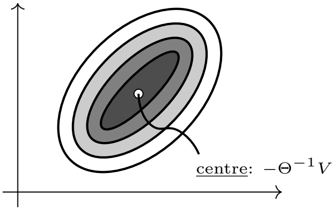
Connections with least squares

Recall that

$$\Sigma_e = \left\{ \Delta \in \mathbb{R}^{n+n\times n} : \left(\Delta + \Theta^{-1}V\right)^\top \Theta \left(\Delta + \Theta^{-1}V\right) \preceq L \right\}$$

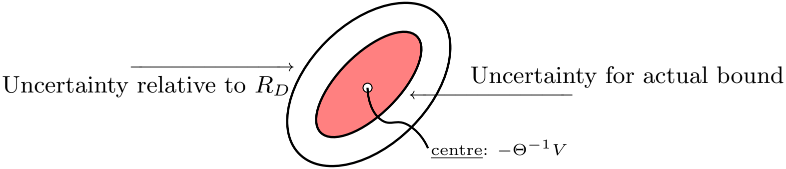
where:

- $\Delta := \begin{bmatrix} B & A \end{bmatrix}^\top$
- $\Theta := W_0 W_0^\top$
- $V := -W_0 X_1^\top$
- $L := V^\top \Theta^{-1} V - \Xi$
- $\Xi := X_1 X_1^\top - R_D R_D^\top$



The ellipsoid centre $-\Theta^{-1}V$ is the least-squares estimate of $(A_\star, B_\star)$ and is purely data-dependent. The size of the uncertainty depends on L, thus on the data *and* the priors (R_D).

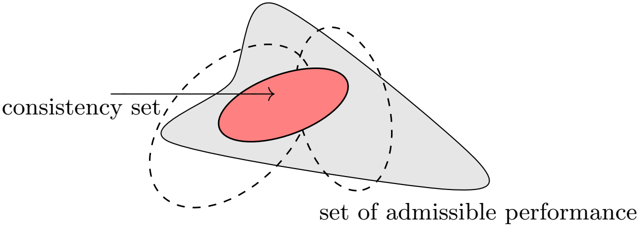
The least-squares estimate has a privileged position and this explains why certainty-equivalence works well in some cases



Detour: Optimal control

Note that problems involving optimal control (like LQR design) are tricky since ”blind” robust formulations need not ensure a sufficient performance level (even unknown sub-optimality gap).

We need the consistency set included in the set of admissible performance



Ideal LQR formulation with unitary weights (Lecture 1):

$$
\begin{aligned}
&\mathrm{minimize}_{Y,P,L}\ \operatorname{trace}(P) + \operatorname{trace}(L)\\
&\text{subject to}\\
&\begin{cases}
\underbrace{(X_1 - D_0)YP^{-1}Y^\top (X_1 - D_0)^\top - P + I_n}_{\mathcal{L}(Y,P,D_0)} \preceq 0\\
P \succeq I_n\\
L - U_0 Y P^{-1} Y^\top U_0^\top \succeq 0\\
X_0 Y = P
\end{cases}
\end{aligned}
$$

which cannot be implemented as D_0 is unknown. Replacing the constraint $\mathcal{L}(Y, P, D_0)$ with $\mathcal{L}(Y, P, D)$ for all $D \in \mathcal{D}_e$ favours too much robustness to the detriment of performance.

We instead look for a solution that trades off robustness for performance via a soft constraint.

Soft constraint formulation:[1]

$$
\begin{aligned}
&\text{minimize}_{Y,P,L}\ \operatorname{trace}(P) + \operatorname{trace}(L) + \alpha \operatorname{trace}(V) \\
&\text{subject to} \\
&\begin{cases}
X_1 Y P^{-1} Y^\top X_1^\top - P + I_n \preceq 0 \\
P \succeq I_n \\
L - U_0 Y P^{-1} Y^\top U_0^\top \succeq 0 \\
X_0 Y = P \\
V - Y P^{-1} Y^\top \succeq 0
\end{cases}
\end{aligned}
$$

where $\alpha > 0$.

- $\alpha \ll 1$ favours performance (close to ideal formulation)
- $\alpha \gg 1$ favours robustness as it favours solutions with $\operatorname{trace}(V) \ll 1$, and this makes $(X_1 - D_0) Y P^{-1} Y^\top (X_1 - D_0)^\top - P + I_n \preceq 0$ easier to satisfy.

[1] C De Persis, P Tesi. Low-complexity learning of linear quadratic regulators from noisy data. Automatica 2021

Instantaneous bounds

Thus far we have considered energy models.

Another important model constrains the admissible disturbances in terms of their 'instantaneous' amplitude. Disturbances in this class are typically called *unknown-but-bounded (UBB)*.

Disturbance model (UBB)

$$\mathcal{D}_i := \left\{ d \in \mathbb{R}^n : \|d\|^2 \leq \epsilon \text{ for known } \epsilon \right\}$$

Note We consider the 2-norm but other norms can be used.

This model is just as important as the energy model but is considerably more difficult to handle. The typical approach is indeed to convert it to an energy model.

We discuss an approach that works directly with instantaneous bounds and results in a tractable formulation (LMI).

Consistency set associated with $\mathcal{D}_i$

Consider again the system

$$x(k+1) = A_\star x(k) + B_\star u(k) + d(k)$$

where $x, d \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Assume now instantaneous bounds for the disturbance, namely $\mathcal{D}_i = \{d \in \mathbb{R}^n : \|d\|^2 \leq \epsilon \text{ for known } \epsilon\}$.

Consistency set

$$\Sigma_i := \bigcap_{k=0}^{T-1} \Sigma_i^{(k)}$$

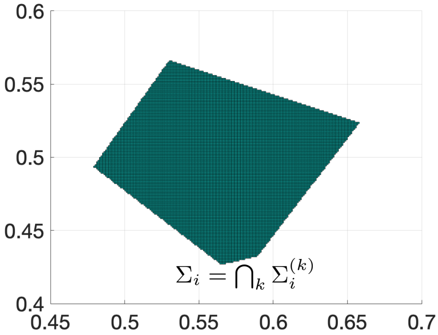
where

$$\Sigma_i^{(k)} := \{(A, B) : x(k+1) = Ax(k) + Bu(k) + d \ \text{ for some } d \in \mathcal{D}_i\}$$

(each samples defines a constraint)

The consistency set now appears as an intersection of sets. The design is more involved as Σ_i has a complicated structure.

Example System $A_\star = B_\star = 0.5$. Set Σ_i resulting from an experiment of length $T = 100$ where $|u(k)| \leq 1$ and $|d(k)| \leq 1$ for all k, both randomly generated.



Converting inst. bounds into energy bounds?

A simple approach to treat instantaneous bounds is to convert them into energy bounds and use previous tools (Petersen's lemma). The procedure is extremely simple:

1. Start with $\mathcal{D}_i = \{d : \|d\|^2 \leq \epsilon\}$
2. Define $\overline{\mathcal{D}}_e := \{D : DD^\top \preceq T\epsilon I_n\}$
3. Stabilize $\overline{\Sigma}_e := \{(A, B) : X_1 = AX_0 + BU_0 + D,\ D \in \overline{\mathcal{D}}_e\}$

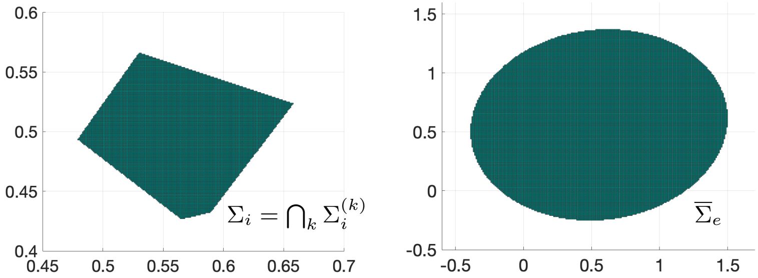
The procedure works because of the following:

Lemma $\Sigma_i \subseteq \overline{\Sigma}_e$ for any experiment of any length T.

On the gap between Σ_i and $\overline{\Sigma}_e$

The gap between the two sets can be very large.

Example System as in the previous example. (Left) Set Σ_i associated with $\mathcal{D}_i = \{d : |d| \leq 1\}$. (Right) Set $\overline{\Sigma}_e$ ($\overline{\mathcal{D}}_e = \{D : DD^\top \preceq 100\}$).



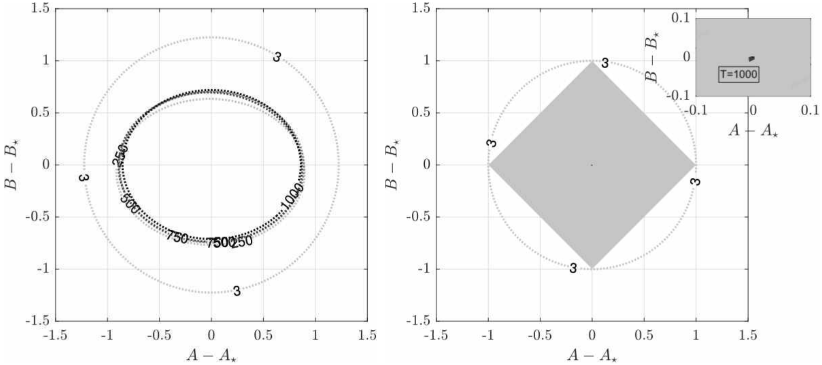
A closer look shows the following:

Lemma [a] Let $\Sigma_i(T)$ be the consistency set associated with an experiment of length T and the disturbance model $\mathcal{D}_i$. Let $\overline{\Sigma}_e(T)$ be the consistency set obtained by converting $\mathcal{D}_i$ into $\overline{\mathcal{D}}_e$. Then:

- $\Sigma_i(T) \subseteq \overline{\Sigma}_e(T)$ (previous lemma)
- $\Sigma_i(T+1) \subseteq \Sigma_i(T)$
- $\overline{\Sigma}_e(T+1) \nsubseteq \overline{\Sigma}_e(T)$

[a] A. Bisoffi, C. De Persis, P. Tesi. Trade-offs in learning controllers from noisy data. Systems & Control Letters 2021

Consistency sets in the previous example for increasing number of data points. (Left) $\overline{\Sigma}_e$. (Right) Σ_i (depicted in grey colors).



A direct tractable approach

As $\overline{\Sigma}_e$ is typically much larger than Σ_i, we want to work directly with Σ_i. The structure of Σ_i is complicated, and we have to search for relaxations. We discuss a relaxation approach that is based on the so-called S-lemma (or S-procedure).

Consistency set

$$\Sigma_i := \bigcap_{k=0}^{T-1} \Sigma_i^{(k)}$$

Problem III

$$\begin{aligned} \text{find} \quad & K, P \succ 0 \\ \text{such that} \quad & (A+BK)P(A+BK)^\top - P \prec 0 \quad \forall (A,B) \in \Sigma_i \end{aligned}$$

Consistency set

$$\Sigma_i := \bigcap_{k=0}^{T-1} \Sigma_i^{(k)}$$

Problem III

$$\begin{aligned} \text{find} \quad & K, P \succ 0 \\ \text{such that} \quad & \underbrace{(A+BK)P(A+BK)^\top - P}_{\mathcal{L}(K,P)} \prec 0 \quad \forall (A,B) \in \Sigma_i \end{aligned}$$

Idea We want to check when one relation $((A,B) \in \Sigma_i)$ implies another one $(\mathcal{L}(K,P) \prec 0)$, where the relation $(A,B) \in \Sigma_i$ consists of a 'system' of quadratic inequalities.

The S-procedure is a relaxation method which tries to solve a system of quadratic inequalities via an LMI relaxation. Very famous because under certain conditions the relaxation is exact or "lossless". [2]

[2] A. Yakubovich. Solution of certain matrix inequalities in the stability theory of nonlinear control systems. Soviet Mathematics Doklady 1962

Main results

Theorem[a] Consider a system $x^+ = A_\star x + B_\star u + d$ with dataset U_0, X_0, X_1. Suppose that each element (column) of the matrix D_0 belongs to $\mathcal{D}_i$ where $\mathcal{D}_i = \left\{ d : \|d\|^2 \leq \epsilon \text{ for known } \epsilon > 0 \right\}$. Problem III is feasible if there exist $P \in \mathbb{R}^{n \times n}, Y \in \mathbb{R}^{m \times n}, \tau_0, \ldots, \tau_{T-1} \in \mathbb{R}_{\geq 0}$ such that

$$\underbrace{\begin{bmatrix} P & 0 & 0 & 0 \\ 0 & -P & -Y^\top & 0 \\ 0 & -Y & 0 & Y \\ 0 & 0 & Y^\top & P \end{bmatrix}}_{Q} - \sum_{k=0}^{T-1} \tau_k \underbrace{\begin{bmatrix} I & x(k+1) \\ 0 & -x(k) \\ 0 & -u(k) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \epsilon I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & x(k+1) \\ 0 & -x(k) \\ 0 & -u(k) \\ 0 & 0 \end{bmatrix}^\top}_{S_k} \succ 0$$

In this case, $K = YP^{-1}$.

[a] A. Bisoffi, C. De Persis, P. Tesi. Trade-offs in learning controllers from noisy data. Systems & Control Letters 2021

Proof (sketch). Recall:

$$\Sigma_i^{(k)} = \left\{(A, B) : x(k+1) = Ax(k) + Bu(k) + d, \|d\|^2 \leq \epsilon\right\}$$

Each constraint can be rewritten as

$$\underbrace{\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \begin{bmatrix} I & x(k+1) \\ 0 & -x(k) \\ 0 & -u(k) \end{bmatrix} \begin{bmatrix} \epsilon I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & x(k+1) \\ 0 & -x(k) \\ 0 & -u(k) \end{bmatrix}^\top \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}}_{\begin{bmatrix} I \\ X \end{bmatrix}^\top S_k \begin{bmatrix} I \\ X \end{bmatrix}} \succeq 0$$

It remains to look at the stability condition.

The stability condition reads

$$(A+BK)P(A+BK)^\top - P \prec 0$$

and it can be rewritten as

$$\underbrace{\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \begin{bmatrix} P & 0 & 0 \\ 0 & -P & -PK^\top \\ 0 & -KP & -KPK^\top \end{bmatrix} \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}}_{\begin{bmatrix} I \\ X \end{bmatrix}^\top Q \begin{bmatrix} I \\ X \end{bmatrix}} \succ 0$$

We want to show that

$$\begin{bmatrix} I \\ X \end{bmatrix}^\top Q \begin{bmatrix} I \\ X \end{bmatrix} \succ 0 \text{ for all } X \text{ such that } \begin{bmatrix} I \\ X \end{bmatrix}^\top S_k \begin{bmatrix} I \\ X \end{bmatrix} \succeq 0 \text{ for all } k = 0, 1, \ldots, T-1$$

By applying the S-lemma, this holds if there exists $\tau_0, \ldots, \tau_{T-1} \in \mathbb{R}_{\geq 0}$ such that

$$Q - \sum_{k=0}^{T-1} \tau_k S_k \succ 0$$

Letting $Y = KP$ and performing a Schur complement the condition above is equivalent to the condition in the statement. $\square$

This approach only provides sufficient conditions. Nonetheless, it is never more conservative than the one that uses $\overline{\Sigma}_e$.

Recall:

Problem III (just solved)

$$\begin{aligned} \text{find} \quad & K, P \succ 0 \\ \text{such that} \quad & (A+BK)P(A+BK)^\top - P \prec 0 \quad \forall (A,B) \in \Sigma_i \end{aligned}$$

Problem I (solvable via Petersen's lemma)

$$\begin{aligned} \text{find} \quad & K, P \succ 0 \\ \text{such that} \quad & (A+BK)P(A+BK)^\top - P \prec 0 \quad \forall (A,B) \in \overline{\Sigma}_e \end{aligned}$$

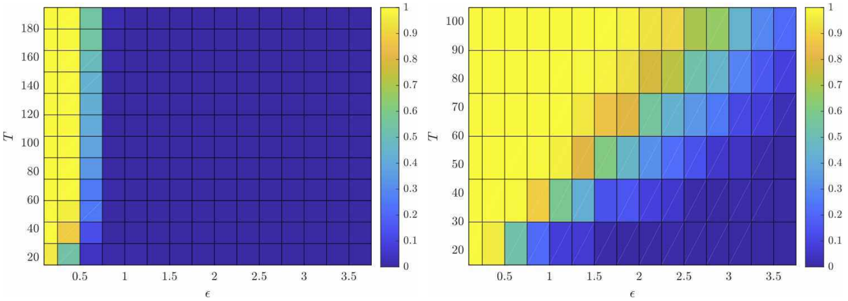
Theorem [a] Problem I $\Longrightarrow$ Problem III.

[a] A. Bisoffi, C. De Persis, P. Tesi. Trade-offs in learning controllers from noisy data. Systems & Control Letters 2021

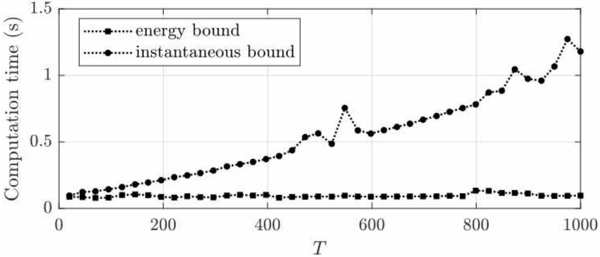
The proof is omitted.

Example

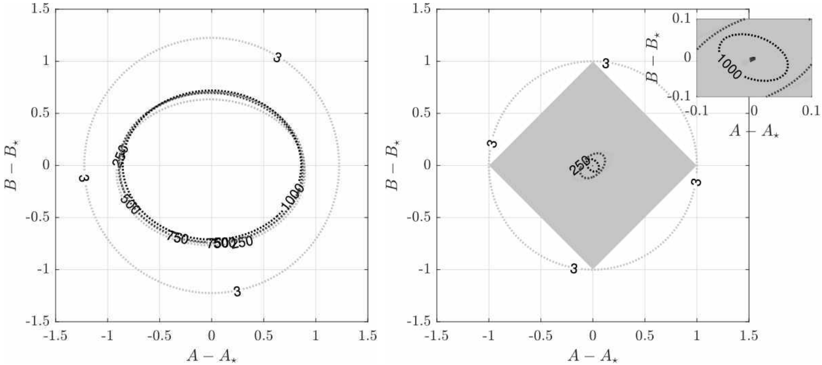
Example for system with 3 states and 2 inputs under different ϵ and T. For each value of ϵ and T, we solve a batch of 100 feasibility problems with $|u| \leq 1$. (Left) Results with $\overline{\Sigma}_e$. (Right) Results with Σ_i.



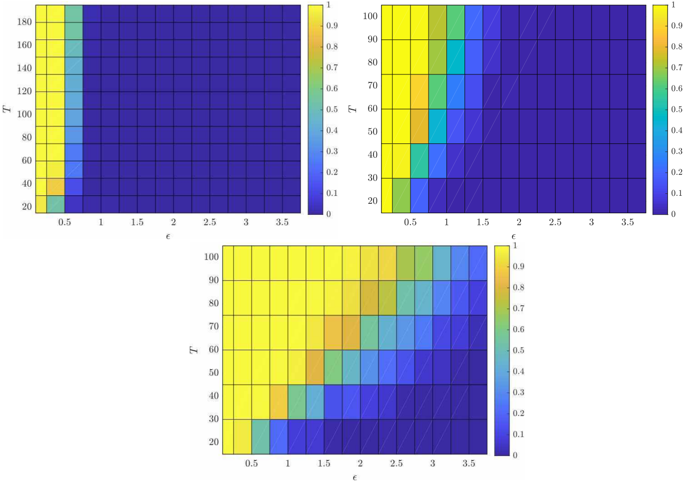
Working with Σ_i increases the computations as we have T more decision variables $(\tau_0, \ldots, \tau_{T-1})$. Except for very large datasets, the price paid in terms of computation time appears negligible.



We can search for other less conservative ellipsoidal over-approximations of Σ_i. However, even for scalar systems finding the ellipsoid of minimum volume that contains Σ_i is NP-complete. In practice, 'good' over-approximations $\overline{\Sigma}_i$ can be found but they can anyway be loose.



Previous example now with a classic approach[3] that determines an outer approximation of the intersection of ellipsoids (top-right figure)



[3] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan. Linear Matrix Inequalities in System and Control Theory (§ 3.7.2), SIAM 1994

Summary and next lecture

- Simple method with stability guarantees, only requires a finite number of data from a low sample-complexity experiment.
- Tight conditions for energy models (ellipsoidal uncertainty)
- Can handle deterministic and stochastic noise, and can be adapted to different noise models (energy and point-wise models).
- Generalizes to robust performance and nonlinear settings.

To follow

Nonlinear input-affine systems, polynomial systems, SOS tools

Selected references

The tight conditions based on Petersen's lemma has been proposed in:

A. Bisoffi, C. De Persis, P. Tesi. Data-driven control via Petersen's lemma. Automatica 2022

The G_K-representation for optimal control (LQR) is treated in:

C. De Persis, P. Tesi. Low-complexity learning of linear quadratic regulators from noisy data. Automatica 2021

The case of instantaneous bounds is discussed in:

A. Bisoffi, C. De Persis, P. Tesi. Trade-offs in learning controllers from noisy data. Systems & Control Letters 2022

Performance via LMI regions:

A. Bisoffi, C. De Persis, P. Tesi. Learning controllers for performance via LMI regions. IEEE Transactions on Automatic Control 2023

Transfer learning for stabilization:

L. Li, C. De Persis, P. Tesi, N. Monshizadeh. Data-based transfer stabilization in linear systems. IEEE Transactions on Automatic Control (under review) 2022

Data driven control

linear and nonlinear systems

Lecture 4

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Nonlinear systems

Nonlinear systems

We consider nonlinear input-affine systems

$$\dot{x} = f_\star(x) + g_\star(x)u$$

where $\underline{f_\star, g_\star}$ are <u>unknown</u> vector fields.

<u>Prior 1</u> $f_\star, g_\star$ can be expressed as linear combinations of basis of known functions

$$f_\star(x) = A_\star Z(x) \quad g_\star(x) = B_\star W(x)$$

with

- $A_\star, B_\star$ unknown constant matrices
- $Z(x) \in \mathbb{R}^N, W(x) \in \mathbb{R}^{M\times m}$ matrices of known functions that encode prior information

Comments

- The linear parametrization allows for a data-dependent representation which is suitable for synthesis and analysis
- We do not allow $Z(x), W(x)$ to depend on parameters, i.e. to have $Z(x,\theta), W(x,\theta)$, with θ unknown, thus compelling the designer to use large number of functions in $Z(x), W(x)$
- The linear parametrization can be replaced by non-parametric models satisfying different priors but the data-dependent representation is more challenging to use for synthesis and analysis

Example

$$\dot{x} = f_\star(x) + g_\star(x)u$$

Priors

- Polynomial system with $f_\star(x)$ of degree ≤ 3 and $g_\star(x)$ of degree 0
- State x of order 2, i.e. $x \in \mathbb{R}^2$

Based on these priors we choose

$$\begin{aligned} Z(x) =& \begin{bmatrix} x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 \end{bmatrix}^\top \\ W(x) =& 1 \end{aligned}$$

Then

$$\dot{x} = A_\star Z(x) + B_\star u \quad \text{with} \quad A_\star \in \mathbb{R}^{2\times 7},\ B_\star \in \mathbb{R}^{2\times 1} \text{ unknown}$$

is the system representation to be used for data-driven control design

Data collection for nonlinear systems

Experiment We run an experiment on the nonlinear system

$$\dot{x} = A_\star Z(x) + B_\star W(x)u + d$$

during which

- An open-loop control u over a finite time interval $\mathcal{I}$ over which a solution $x(t)$ exists is applied
- T triples of samples $\{\dot{x}(t_k), x(t_k), u(t_k)\}_{k=0}^{T-1}$ at sampling times $t_k \in \mathcal{I}$ (not necessarily evenly spaced) are collected
- Process disturbance d affects the dynamics
- Samples $\{\dot{x}(t_k), x(t_k), u(t_k)\}_{k=0}^{T-1}$ are measured with no noise*
- Vectors $\{Z(x(t_k)), W(x(t_k))u(t_k)\}_{k=0}^{T-1}$ are computed with no errors*

At each sampling time t_k

$$\dot{x}(t_k) = A_\star Z(x(t_k)) + B_\star W(x(t_k))u(t_k) + d(t_k) \quad k = 0, 1, \ldots, T-1$$

*Both cases of noisy measurements and computation errors can be included at the price of a more involved analysis

Disturbance model

- <u>Dataset identities</u> The dataset satisfies the identities

$$\begin{aligned}
&\underbrace{\begin{bmatrix}\dot{x}(t_0) & \dot{x}(t_1) & \dots & \dot{x}(t_{T-1})\end{bmatrix}}_{X_1} \\
= \quad & A_\star \underbrace{\begin{bmatrix}Z(x(t_0)) & Z(x(t_1)) & \dots & Z(x(t_{T-1}))\end{bmatrix}}_{Z_0} \\
+ \quad & B_\star \underbrace{\begin{bmatrix}W(x(t_0))u(t_0) & W(x(t_1))u(t_1) & \dots & W(x(t_{T-1}))u(t_{T-1})\end{bmatrix}}_{\overline{U}_0} \\
& + \underbrace{\begin{bmatrix}d(t_0) & d(t_1) & \dots & d(t_{T-1})\end{bmatrix}}_{D_0} \\
= \quad & A_\star Z_0 + B_\star \overline{U}_0 + D_0 = \begin{bmatrix}B_\star & A_\star\end{bmatrix} \begin{bmatrix}\overline{U}_0 \\ Z_0\end{bmatrix} + D_0
\end{aligned}$$

<u>Key fact</u> This identity on data is formally analogous to the one used for linear systems

- <u>Standing assumption</u> rank $\begin{bmatrix}\overline{U}_0 \\ Z_0\end{bmatrix} = M + N$

Data-dependent representation of nonlinear systems

Prior 2 - (Energy constrained disturbances) Given the disturbance sequence $\{d(t_k)\}_{k=0}^{T-1}$, the matrix

$$D_0 = \begin{bmatrix} d(t_0) & d(t_1) & \dots & d(t_{T-1}) \end{bmatrix}$$

belongs to the set

$$\mathcal{D}_e = \left\{ D \in \mathbb{R}^{n \times T} \colon DD^\top \preceq R_D R_D^\top \right\}$$

for some known R_D.

Data-dependent representation of nonlinear systems

- Feasible system matrices set Under the given priors and the data set, the set of feasible system's matrices is given by

$$\mathcal{C} := \left\{ (A,B) \colon \ X_1 = AZ_0 + B\overline{U}_0 + D, \ D \in \mathbb{R}^{n \times T}, \ \ DD^\top \preceq R_D R_D^\top \right\}$$

- By eliminating D in the identity $X_1 = AZ_0 + B\overline{U}_0 + D$, and setting

$$\overline{W}_0 := \begin{bmatrix} \overline{U}_0 \\ Z_0 \end{bmatrix}$$

one obtains the condition

$$\left(X_1 - \begin{bmatrix} B & A \end{bmatrix} \overline{W}_0 \right) \left(X_1 - \begin{bmatrix} B & A \end{bmatrix} \overline{W}_0 \right)^\top \preceq R_D R_D^\top$$

and the equivalent representation

$$\mathcal{C} := \left\{ (A,B) \colon \ \begin{bmatrix} B & A \end{bmatrix} \Theta \begin{bmatrix} B^\top \\ A^\top \end{bmatrix} + \begin{bmatrix} B & A \end{bmatrix} V + V^\top \begin{bmatrix} B^\top \\ A^\top \end{bmatrix} + \Xi \preceq 0 \right\}$$

where

$$\Theta := \overline{W}_0 \overline{W}_0^\top \quad V := -\overline{W}_0 X_1^\top \quad \Xi := X_1 X_1^\top - R_D R_D^\top$$

all known data-dependent matrices

Data-dependent representation of nonlinear systems

- Feasible system matrices set as a matrix ellipsoid The set

$$\mathcal{C} := \left\{ (A,B) \colon \begin{bmatrix} B & A \end{bmatrix} \Theta \begin{bmatrix} B^\top \\ A^\top \end{bmatrix} + \begin{bmatrix} B & A \end{bmatrix} V + V^\top \begin{bmatrix} B^\top \\ A^\top \end{bmatrix} + \Xi \preceq 0 \right\}$$

has the form of a matrix ellipsoid

$$\left\{ Z \colon Z^\top \Theta Z + Z^\top V + V^\top Z + \Xi \preceq 0 \right\}$$

- Feasible system matrices set Bearing in mind that Θ non-singular by standing assumption

$$\mathrm{rank} \overline{W}_0 = M + N \quad \text{with} \quad \overline{W}_0 = \begin{bmatrix} \overline{U}_0 \\ Z_0 \end{bmatrix}$$

the matrix ellipsoid can be rearranged in the form

$$\mathcal{C} = \left\{ (A,B) \colon \begin{bmatrix} B & A \end{bmatrix}^\top = -\Theta^{-1} V + \Theta^{-1/2} Y L^{1/2}, \ \|Y\| \leq 1 \right\}$$

with

$$-\Theta^{-1} V = (X_1 \overline{W}_0^\dagger)^\top \quad L = X_1 (\overline{W}_0^\top (\overline{W}_0 \overline{W}_0^\top)^{-1} \overline{W}_0 - I) X_1^\top + R_D R_D^\top \succeq 0$$

- $\mathcal{C}$ bounded $\| \begin{bmatrix} B & A \end{bmatrix}^\top \| \leq \|\Theta^{-1} V\| + \lambda_{\min}(\Theta)^{-1/2} \|L^{1/2}\|$ for any $(A,B) \in \mathcal{C}$

Data-dependent representation of nonlinear systems

Summary Nonlinear input-affine systems

$$\dot{x} = f_\star(x) + g_\star(x)u + d$$

Prior 1 Linear parametrization of $f_\star, g_\star$

$$f_\star(x) = A_\star Z(x) \quad g_\star(x) = B_\star W(x)$$

Data set T-long data set $\{\dot{x}(t_k), x(t_k), u(t_k)\}_{k=0}^{T-1}$ satisfying

$$\dot{x}(t_k) = A_\star Z(x(t_k)) + B_\star W(x(t_k))u(t_k) + d(t_k) \quad k = 0, 1, \dots, T-1$$

Prior 2 Energy constrained disturbances

$$\left\{d(t_0), d(t_1), \dots, d(t_{T-1})\right\} \quad \text{such that} \quad \sum_{k=0}^{T-1} d(t_k)(t_k)^\top \preceq R_D R_D^\top \quad \text{for some } R_D$$

Data-dependent representation of nonlinear systems

The nonlinear input-affine system

$$\dot{x} = f_\star(x) + g_\star(x)u + d$$

belongs to the set of data-dependent representations

$$\dot{x} = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} W(x)u \\ Z(x) \end{bmatrix} + d \quad \text{with} \quad (A, B) \in \mathcal{C}$$

where

$$\mathcal{C} = \left\{ (A, B) \colon \begin{bmatrix} B & A \end{bmatrix}^\top = -\Theta^{-1} V + \Theta^{-1/2} Y L^{1/2},\ \|Y\| \leq 1 \right\}$$

Control synthesis

Towards data-driven control design for nonlinear systems

Data enable the replacement of the <u>unknown</u> system

$$\dot{x} = A_\star Z(x) + B_\star W(x)u + d$$

with the <u>uncertain</u> system

$$\dot{x} = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} W(x)u \\ Z(x) \end{bmatrix} + d \quad \text{with} \quad (A, B) \in \mathcal{C}$$

There are <u>manifold challenges</u> to the synthesis of a controller for this data-dependent representation

- There is no apparent structure in the system to exploit
- The system is nonlinear and uncertain
- For stabilization purposes, both a Lyapunov function and a controller must be designed

Towards data-driven control design for nonlinear systems

Data enable the replacement of the <u>unknown</u> system

$$\dot{x} = A_\star Z(x) + B_\star W(x)u + d$$

with the <u>uncertain</u> system

$$\dot{x} = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} W(x)u \\ Z(x) \end{bmatrix} + d \quad \text{with} \quad (A, B) \in \mathcal{C}$$

- We focus on <u>stabilization</u>, the quintessential control problem
- We assume that the equilibrium pair (x_{eq}, u_{eq}) of interest is known and equal to the origin

Towards data-driven control design for nonlinear systems

Main idea to exploit the linear parametrization of the system

- Consider the controller

$$u = K(x)Z(x) \quad \text{with } K(x) \text{ matrix of functions}$$

to obtain the closed-loop system

$$\dot{x} = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} W(x)K(x) \\ I_N \end{bmatrix} Z(x) + d \quad \text{with} \quad (A,B) \in \mathcal{C}$$

- Consider the Lyapunov function

$$V(x) = Z(x)^\top P Z(x) \quad \text{with } P \succ 0$$

- Look for conditions under which there exist $P \succ 0$ and $K(x)$ such that

$$\dot{V}(x) = 2Z(x)^\top P \frac{\partial Z(x)}{\partial x} \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} W(x)K(x) \\ I_N \end{bmatrix} Z(x) < 0 \quad \text{for all } (A,B) \in \mathcal{C}$$

We will be soon more specific on the domain of validity of the Lyapunov inequality

A remark on the robustness of the closed-loop system

- Asymptotic stability of the nonlinear closed-loop system

$$\dot{x} = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} W(x)K(x) \\ I_N \end{bmatrix} Z(x) + d \quad \text{with} \quad (A, B) \in \mathcal{C}$$

  for $d = 0$ does not guarantee its robustness with respect to arbitrary $d \neq 0$

- However, if the equilibrium is globally asymptotically stable then the system is robust to sufficiently small disturbances, namely*

$\exists\, \beta(\cdot,\cdot) \in \mathcal{KL}$, $\gamma(\cdot) \in \mathcal{K}$, $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, continuous with $\sigma(s) \neq 0$ for $s \neq 0$, such that $\forall x(0), d(\cdot)$ for which $\|d(\cdot)\|_\infty \leq \sigma(\|x(0)\|)$

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\|d(\cdot)\|_\infty) \quad \forall t \geq 0$$

We focus on a globally stabilizing controller and neglect d

*E.D. Sontag. "Further facts about input to state stabilization", *IEEE Transactions on Automatic Control*, 35(4), 473–476, 1990.

Towards data-driven control design for nonlinear systems

- $Z(x)$ is used in the representation of $f_\star(x)$ and may be high dimensional, leading to higher computational load in the design phase. For feedback control u and Lyapunov function V a much lower-dimensional $\hat{Z}(x)$ might suffice, related to $Z(x)$ via

$$Z(x) = H(x)\hat{Z}(x) \quad \text{with } H(x) \text{ matrix of functions}$$

- Feedback control and Lyapunov function A revised controller

$$u = K(x)\hat{Z}(x) \quad \text{with } K(x) \text{ matrix of functions}$$

and a revised Lyapunov function

$$V(x) = \hat{Z}(x)^\top P \hat{Z}(x) \quad \text{with } P \succ 0$$

- Properties of $\hat{Z}(x)$

$$\begin{aligned}
&\hat{Z}(x) = 0 \Leftrightarrow x = 0 \ (\Rightarrow V(x) \text{ globally positive definite})\\
&\hat{Z}(x) \text{ radially unbounded } (\Rightarrow V(x) \text{ radially unbounded})\\
&\hat{Z}(x) \text{ contains the state vector } x
\end{aligned}$$

Example

$$\dot{x} = f_\star(x) + g_\star(x)u$$

Priors

- Polynomial system with $f_\star(x)$ of degree ≤ 3 and $g_\star(x)$ of degree 0
- State x of order 2, i.e. $x \in \mathbb{R}^2$

Previously we chose

$$\begin{aligned} Z(x) =& \begin{bmatrix} x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 \end{bmatrix}^\top \\ W(x) =& 1 \end{aligned}$$

If now we choose

$$\hat{Z}(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top$$

then

$$H(x) = \begin{bmatrix} 1 & 0 & x_2 & x_1 & 0 & x_1x_2 & 0 & x_1^2 & 0 \\ 0 & 1 & 0 & 0 & x_2 & 0 & x_1x_2 & 0 & x_2^2 \end{bmatrix}^\top$$

$H(x)$ not unique

Data-driven control design of nonlinear systems

- Write the closed-loop system...

$$\dot{x} = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} W(x)u \\ Z(x) \end{bmatrix} \quad \text{with} \quad (A,B) \in \mathcal{C} \quad u = K(x)\hat{Z}(x)$$

as

$$\dot{x} = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} W(x)K(x) \\ H(x) \end{bmatrix} \hat{Z}(x)$$

- ...and the Lyapunov inequality

$$\dot{V}(x) = 2\hat{Z}(x)^\top P \frac{\partial \hat{Z}(x)}{\partial x} \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} W(x)K(x) \\ H(x) \end{bmatrix} \hat{Z}(x) < 0 \quad \forall x \neq 0$$
$$\text{for all } (A,B) \in \mathcal{C}$$

Observe the product of the two decision variables P and $K(x)$

Data-driven control design of nonlinear systems

To prevent the product of the two decision variables P and $K(x)$

1) Pull the vector $P\hat{Z}(x)$ out of the Lyapunov inequality

$$\dot{V}(x) = \quad 2\hat{Z}(x)^\top P \frac{\partial \hat{Z}(x)}{\partial x} \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} W(x)K(x)P^{-1} \\ H(x)P^{-1} \end{bmatrix} P\hat{Z}(x)$$

2) Perform a <u>change of variables</u> analogous to the case of linear systems

$$F(x) := K(x)P^{-1} \quad Q = P^{-1}$$

which yields

$$\dot{V}(x) = \quad \hat{Z}(x)^\top P \, M(x) \, P\hat{Z}(x)$$

where

$$M(x) = \frac{\partial \hat{Z}(x)}{\partial x} \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix} + \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top \begin{bmatrix} B & A \end{bmatrix}^\top \frac{\partial \hat{Z}(x)}{\partial x}^\top$$

Data-driven control design of nonlinear systems

Express Lyapunov stability conditions in terms of the matrix $M(x)$

$$\exists Q \succ 0,\ F(x):\ M(x) \prec 0 \quad \forall x \neq 0 \quad \Longrightarrow \quad \dot{V}(x) < 0 \quad \forall x \neq 0$$

The advantage of working with

$$M(x) = \frac{\partial \hat{Z}(x)}{\partial x} \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix} + \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top \begin{bmatrix} B & A \end{bmatrix}^\top \frac{\partial \hat{Z}(x)}{\partial x}^\top$$

is that the decision variables appear linearly, at the price of some conservatism

Data-driven control design of nonlinear systems

Problem Find matrix $Q \succ 0$ and matrix of functions $F(x)$ such that

$$\frac{\partial \hat{Z}(x)}{\partial x} \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix} + \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top \begin{bmatrix} B & A \end{bmatrix}^\top \frac{\partial \hat{Z}(x)}{\partial x}^\top \prec 0$$

for all $x \neq 0$ and all $(A, B) \in \mathcal{C}$

If $F(x)$ and $Q \succ 0$ are found, then (recall that $u = K(x)\hat{Z}(x)$ and $K(x)Q = F(x)$)

$$u = F(x)Q^{-1}\hat{Z}(x)$$

is a globally asymptotically stabilizing controller and

$$V(x) = \hat{Z}(x)^\top Q^{-1}\hat{Z}(x)$$

is a Lyapunov function

Data-driven control design of nonlinear systems

Problem Find matrix $F(x)$ and matrix $Q \succ 0$ such that

$$M(x) \prec 0 \quad \text{for all } x \neq 0 \text{ and all } (A, B) \in \mathcal{C}$$

where

$$M(x) = \frac{\partial \hat{Z}(x)}{\partial x} \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix} + \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top \begin{bmatrix} B & A \end{bmatrix}^\top \frac{\partial \hat{Z}(x)}{\partial x}^\top$$

Bearing in mind that $(A, B) \in \mathcal{C}$ is equivalent to

$$\begin{bmatrix} B & A \end{bmatrix}^\top = Z_c + U^{-1/2} Y L^{1/2} \quad \|Y\| \leq 1$$

where $\underline{Z_c = -\Theta^{-1} V \text{ for short}}$, we write

$$\begin{aligned} M(x) = \quad & \frac{\partial \hat{Z}(x)}{\partial x} Z_c^\top \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix} + \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top Z_c \frac{\partial \hat{Z}(x)}{\partial x}^\top \\ & + \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top U^{-1/2} Y L^{1/2} \frac{\partial \hat{Z}(x)}{\partial x}^\top + (\star)^\top \end{aligned}$$

Data-driven control design of nonlinear systems

Let us single out the different parts of $M(x)$

$$M(x) = \overbrace{\frac{\partial \hat{Z}(x)}{\partial x} Z_c^\top \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix} + \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top Z_c \frac{\partial \hat{Z}(x)}{\partial x}^\top}^{\mathcal{G}(x)} + \underbrace{\begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top U^{-1/2}}_{\mathcal{M}(x)} \underbrace{Y}_{Y} \underbrace{L^{1/2} \frac{\partial \hat{Z}(x)}{\partial x}^\top}_{\mathcal{N}(x)} + (\star)^\top$$

By pointwise application of Petersen's lemma

$$M(x) \prec 0 \quad \forall Y : Y^\top Y \preceq I$$

$$\Updownarrow$$

$$\exists \varepsilon(x) > 0 \colon \mathcal{G}(x) + \varepsilon(x)\mathcal{M}(x)\mathcal{M}(x)^\top + \varepsilon(x)^{-1}\mathcal{N}(x)^\top\mathcal{N}(x) \prec 0$$

Data-driven control design of nonlinear systems

By writing explicitly the condition

$$\mathcal{G}(x) + \varepsilon(x)\mathcal{M}(x)\mathcal{M}(x)^\top + \varepsilon(x)^{-1}\mathcal{N}(x)^\top\mathcal{N}(x) \prec 0$$

we obtain

$$\begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top Z_c \frac{\partial \hat{Z}(x)}{\partial x}^\top + (\star)^\top + \varepsilon(x) \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top \Theta^{-1} \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}$$
$$\varepsilon(x)^{-1} \frac{\partial \hat{Z}(x)}{\partial x} L \frac{\partial \hat{Z}(x)}{\partial x}^\top \prec 0$$

By pointwise Schur complement, the latter is equivalent to

$$\begin{bmatrix} \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top Z_c \frac{\partial \hat{Z}(x)}{\partial x}^\top + (\star)^\top + \varepsilon(x)^{-1} \frac{\partial \hat{Z}(x)}{\partial x} L \frac{\partial \hat{Z}(x)}{\partial x}^\top & \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top \\ \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix} & -\varepsilon(x)^{-1}\Theta \end{bmatrix} \prec 0$$

Recap – what we understood so far

Under

$$\text{Prior 1} \quad \dot{x} = f_\star(x) + g_\star(x)u = A_\star Z(x) + B_\star W(x)u$$

$$\text{Prior 2} \quad \{d(t_k)\}_{k=0}^{T-1} \in \left\{ \{d^k\}_{k=0}^{T-1} : \sum_{k=0}^{T-1} d^k {d^k}^\top \preceq R_D R_D^\top \right\}$$

let $\hat{Z}(x), H(x)$ be matrices of functions such that

$$Z(x) = H(x)\hat{Z}(x) \quad \text{with} \quad \hat{Z}(x) = \begin{bmatrix} x^\top & \dots \end{bmatrix}^\top$$

If there exist $Q \succ 0$, matrix $F(x)$ and $\mu(x) > 0$ such that

$$\begin{bmatrix} \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top Z_c \dfrac{\partial \hat{Z}(x)}{\partial x}^\top + (\star)^\top + \mu(x) \dfrac{\partial \hat{Z}(x)}{\partial x} L \dfrac{\partial \hat{Z}(x)}{\partial x}^\top & \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top \\ \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix} & -\mu(x)\Theta \end{bmatrix} \prec 0$$

for all $x \neq 0$, then

$$\begin{aligned} u &= F(x)Q^{-1}\hat{Z}(x) && \text{globally asymptotically stabilizer} \\ V(x) &= \hat{Z}(x)^\top Q^{-1}\hat{Z}(x) && \text{Lyapunov function} \end{aligned}$$

Comments

- The previous condition requires the following to hold

$$\frac{\partial \hat{Z}(x)}{\partial x} Z_c^\top \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix} + (\star)^\top + \varepsilon(x)^{-1} \frac{\partial \hat{Z}(x)}{\partial x} L \frac{\partial \hat{Z}(x)}{\partial x}^\top \prec 0$$

  That is, the controller matrix $F(x)$ must stabilize the system at the centre of the matrix ellipsoid

$$\dot{x} = Z_c^\top \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix} = A_{LS}Z(x) + B_{LS}W(x)u \text{ with } u = F(x)\hat{Z}(x)$$

  with some stability margin: $\dot{V}(x) < -\|\frac{\partial \hat{Z}(x)}{\partial x}^\top PZ(x)\|_L^2$

- For each fixed $x \neq 0$, the condition is an LMI in the variables $Q \succ 0$ and $F(x)$. If solvable for each $x \neq 0$ in real time, then we would have a numerical solution to the stabilization problem.

- Major issues no analytic solution; few insights in the properties of feedback u; any further analysis of robustness, performance etc. is hard; feasibility of the solution for each $x \neq 0$ uncheckable a priori

- To obtain a tractable solution we strengthen the priors about the unknown system

Nonlinear polynomial systems

We consider nonlinear input-affine systems

$$\dot{x} = f_\star(x) + g_\star(x)u$$

where $\underline{f_\star, g_\star}$ are unknown polynomial vector fields.

Prior 1 - revised $f_\star, g_\star$ can be expressed as linear combinations of basis of known functions

$$f_\star(x) = A_\star Z(x) \quad g_\star(x) = B_\star W(x)$$

with

- $A_\star, B_\star$ unknown constant matrices
- $Z(x) \in \mathbb{R}^N, W(x) \in \mathbb{R}^{M\times m}$ matrices of all distinct monomials that appear in $f_\star(x), g_\star(x)$
- If such a granular prior on $f_\star(x), g_\star(x)$ is not available, then we assume an upper bound on the degrees of $f_\star(x), g_\star(x)$ and $Z(x), W(x)$ will consists of a basis for such polynomials

Why polynomial systems?

- Computational advantage The analysis and design of polynomial control systems rests on Sum-of-Squares (SOS) programming, which can be eventually reduced to LMIs
- Universal approximators Polynomial functions are universal approximators (Stone-Weierstrass theorem), hence learning control for polynomial systems might enable learning control for general nonlinear systems.
- Carleman linearization A special class of polynomial systems are the bilinear systems used in nonlinear control design via Carleman linearization
- Analysis extendible to other classes of nonlinear systems that include e.g. trigonometric and rational functions via addition of new state variables and (in)equality constraints (Anderson, Chesi, Hancock, Papachristodolou, Peet, Tedrake, Valmorbida,...)

Sum of Squares polynomials

A polynomial $p : \mathbb{R}^n \to \mathbb{R}$ is positive if $p(x) \geq 0$ for all x

---

A polynomial $p : \mathbb{R}^n \to \mathbb{R}$ is a sum of squares (SOS) polynomial if

$$p(x) = \sum_i p_i(x)^2 \quad \text{for some polynomials } p_i(x)$$

$p(x)$ SOS $\Rightarrow$ $p(x)$ positive (but not the converse)

Establishing whether a polynomial is an SOS is a convex optimization problem and provides an efficient method to certify the positivity of a polynomial

P.A. Parrilo, "Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization". Ph.D. dissertation, California Institute of Technology, Pasadena, California, 2000.
G. Chesi, "LMI techniques for optimization over polynomials in control: A survey." IEEE Transactions on Automatic Control, 55, 11, pp. 2500-2510, 2010.

Sum of Squares polynomials

A polynomial $p : \mathbb{R}^n \to \mathbb{R}$ of degree $2d$ can always be represented via its square matricial representation, which gives an easy way to test the SOS property

Theorem The polynomial p of degree $2d$ is an SOS if and only if there exists a symmetric matrix $L \succeq 0$ such that

$$p(x) = \zeta(x)^\top L \zeta(x)$$

where $\zeta(x)$ is the vector of all distinct monomials of degree less than or equal to d.

Proof If $p(x) = \zeta(x)^\top L \zeta(x)$ with $L \succeq 0$, then by Cholesky factorization $L = V^\top V$ and therefore $p(x) = (V\zeta(x))^2$, i.e. $p(x)$ is an SOS.

Only if By definition of SOS polynomial, $p(x) = \sum_i p_i(x)^2$. Each $p_i(x)$ is a polynomial of at most degree d, hence there exists c_i such that $p_i(x) = c_i^\top \zeta(x)$. It follows that

$$\begin{aligned} p(x) = & \sum_i p_i(x)^2 = \sum_i (c_i^\top \zeta(x))^2 = \sum_i \zeta(x)^\top c_i c_i^\top \zeta(x) \\ = & \zeta(x)^\top \left(\textstyle\sum_i c_i c_i^\top \right) \zeta(x) =: \zeta(x)^\top L \zeta(x) \quad \blacksquare \end{aligned}$$

Sum of Squares polynomials

Computational procedure

Given a polynomial $p : \mathbb{R}^n \to \mathbb{R}$ of degree $2d$ and the vector $\zeta(x) \in \mathbb{R}^N$, we seek the $N(N+1)/2$ entries of L that solve the equation

$$p(x) = \zeta(x)^\top L \zeta(x)$$

Example $p(x) = g_4 x^4 + g_3 x^3 + g_2 x^2 + g_1 x + g_0$, with $x \in \mathbb{R}$. In this case, $n = 1$ and $2d = 4$ and

$$\zeta(x) = \begin{bmatrix} 1 & x & x^2 \end{bmatrix}^\top$$

Hence

$$\zeta(x)^\top L \zeta(x) = \ell_{33} x^4 + 2\ell_{23} x^3 + (2\ell_{13} + \ell_{22}) x^2 + \ell_{12} x + \ell_{11} \quad L = [\ell_{ij}]$$

Matching the coefficients on both sides of $p(x) = \zeta(x)^\top L \zeta(x)$ returns the equations

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ell_{11} \\ \ell_{12} \\ \ell_{13} \\ \ell_{22} \\ \ell_{23} \\ \ell_{33} \end{bmatrix} = \begin{bmatrix} g_4 \\ g_3 \\ g_2 \\ g_1 \\ g_0 \end{bmatrix} \Leftrightarrow L = \begin{bmatrix} g_0 & g_1/2 & 0 \\ g_1/2 & g_2 & g_3/2 \\ 0 & g_3/2 & g_4 \end{bmatrix} + \ell_{13} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Sum of Squares polynomials

Example (cont'd) The problem becomes

$$\begin{array}{rl} \text{Find} & \ell_{13} \\ \text{such that} & \begin{bmatrix} g_0 & g_1/2 & 0 \\ g_1/2 & g_2 & g_3/2 \\ 0 & g_3/2 & g_4 \end{bmatrix} + \ell_{13} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \succeq 0 \quad \blacksquare \end{array}$$

Instead of manually performing these steps, the task of checking whether a given polynomial p is an SOS is automatically performed by software packages such as SOSTOOLS

Sum of Squares polynomial matrices

A polynomial matrix $P : \mathbb{R}^n \to \mathbb{R}^{r\times r}$ is positive if it is symmetric and $P(x) \succeq 0$ for all x

---

A polynomial matrix $P : \mathbb{R}^n \to \mathbb{R}^{r\times r}$ is an SOS polynomial matrix if

$$P(x) = \textstyle\sum_i P_i(x)^\top P_i(x) \quad \text{for some polynomial matrices } P_i(x) \text{ (not necessarily square)}$$

Let $P_i(x) : \mathbb{R}^n \to \mathbb{R}^{q\times r}$ have degree d and

$$\zeta(x) = \begin{bmatrix}\zeta_1(x) \dots \zeta_N(x)\end{bmatrix}^\top$$

be the vector of all distinct monomials of degree less than or equal to d. Then

$$\begin{aligned} P_i(x) &= \quad C_{i1}\zeta_1(x) + \ldots + C_{iN}\zeta_N(x) \\ &= \quad \begin{bmatrix} C_{i1} & \dots & C_{iN} \end{bmatrix} \begin{bmatrix} \zeta_1(x) I_r \\ \vdots \\ \zeta_N(x) I_r \end{bmatrix} = C_i(\zeta(x) \otimes I_r) \quad C_{ij} \in \mathbb{R}^{q\times r} \end{aligned}$$

$P(x)$ SOS $\Rightarrow$ $P(x)$ positive (but not the converse)

Sum of Squares polynomials

Theorem The polynomial matrix $P : \mathbb{R}^n \to \mathbb{R}^{r\times r}$ of degree $2d$ is an SOS if and only if there exists a symmetric matrix $L \succeq 0$ such that

$$P(x) = (\zeta(x) \otimes I_r)^\top L(\zeta(x) \otimes I_r)$$

where $\zeta(x)$ is the vector of all distinct monomials of degree less than or equal to d.

Proof. If $P(x) = (\zeta(x) \otimes I_r)^\top L(\zeta(x) \otimes I_r)$ with $L \succeq 0$, then by Cholesky factorization $L = V^\top V$ and therefore $P(x) = P_1(x)^\top P_1(x)$, with $P_1(x) = V(\zeta(x) \otimes I_r)$.

Only if By definition of SOS polynomial matrix, $P(x) = \sum_i P_i(x)^\top P_i(x)$. Each $P_i(x)$ is a polynomial of at most degree d, hence there exists C_i such that $P_i(x) = C_i(\zeta(x) \otimes I_r)$. It follows that

$$\begin{aligned} P(x) = &\sum_i P_i(x)^\top P_i(x) = \sum_i (\zeta(x) \otimes I_r)^\top C_i^\top C_i(\zeta(x) \otimes I_r) \\ = &(\zeta(x) \otimes I_r)^\top \sum_i C_i^\top C_i(\zeta(x) \otimes I_r) =: (\zeta(x) \otimes I_r)^\top L(\zeta(x) \otimes I_r) \quad \blacksquare \end{aligned}$$

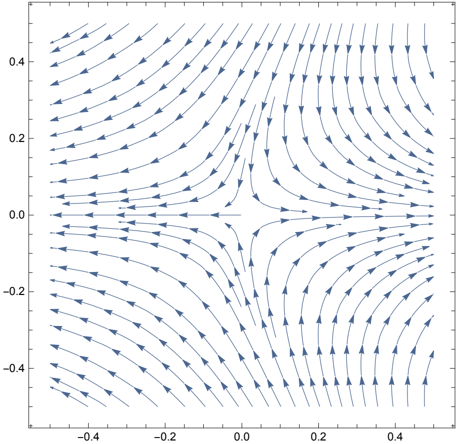
An example

Consider the following example°

$$\begin{aligned}\dot{x} &= f_\star(x) + g_\star(x)u \\ &= \begin{bmatrix} x_1 - 2x_2^2 \\ -x_2 - x_1x_2 - 2x_2^3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u\end{aligned}$$

The origin is an unstable unforced ($u = 0$) equilibrium of the system. We would like to

- stabilise
- find a Lyapunov function



° Modified from Example 7.2 in P.A. Parrilo, "Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization". Ph.D. dissertation, California Institute of Technology, Pasadena, California, 2000.

An example

$$\begin{aligned}\dot{x} &= f_\star(x) + g_\star(x)u \\ &= \begin{bmatrix} x_1 - 2x_2^2 \\ -x_2 - x_1x_2 - 2x_2^3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u\end{aligned}$$

The origin is an unstable unforced ($u = 0$) equilibrium of the system. We would like to

- stabilise the equilibrium
- find a Lyapunov function

Here

$$Z(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_1x_2 \\ x_2^2 \\ x_2^3 \end{bmatrix} \qquad W(x) = 1$$

Hence

$$\begin{aligned}A_\star &= \begin{bmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 & -2 \end{bmatrix} \\ B_\star &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}\end{aligned}$$

Example

We look for

$$\begin{aligned} \text{controllers} \quad & u = K(x)\hat{Z}(x) \\ \text{Lyapunov functions} \quad & V(x) = \hat{Z}(x)^\top P \hat{Z}(x) \end{aligned}$$

with

$$\hat{Z}(x) = x$$

Hence

$$Z(x) = H(x)\hat{Z}(x) \quad \text{with} \quad Z(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_1 x_2 \\ x_2^2 \\ x_2^3 \end{bmatrix}, \quad H(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ x_2 & 0 \\ 0 & x_2 \\ 0 & x_2^2 \end{bmatrix}$$

To achieve stability, we impose

$$\frac{\partial \hat{Z}(x)}{\partial x} \begin{bmatrix} B_\star & A_\star \end{bmatrix} \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix} + \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top \begin{bmatrix} B_\star & A_\star \end{bmatrix}^\top \frac{\partial \hat{Z}(x)}{\partial x}^\top \prec 0$$

where $F(x) = K(x)P^{-1}$, $Q = P^{-1}$

SOSTOOLS script

```
% Defining the state variables x1 x2 x
syms x1 x2 real
x = [x1,x2];

% Defining Astar, Bstar
Astar = [1 0 0 -2 0 ; 0 -1 -1 0 -2];
Bstar = [1 ; 0];

% System dimensions
n=2; m=1;
sizeAstar=size(Astar); sizeBstar=size(Bstar);
M=sizeBstar(1,2);

% Vector hat_Z(x)=x appearing in the Lyapunov function
hat_Z = [x1;x2];
hat_N=length(hat_Z);

% Jacobian of hat_Z(x)
d_hat_Z=jacobian(hat_Z);

%Matrix H(x) relating Z(x) and hat_Z(x)
```

SOSTOOLS script

```
% Defining the matrix Q appearing in the Lyapunov function
% as a polynomial matrix of degree 0
degreeQ=0;
[prog,Q]=sospolymatrixvar(prog,monomials(x,degreeQ),
                      [hat_N,hat_N],'symmetric');

% Constraint on Q to be an SOS matrix
prog = sosmatrixineq (prog, Q,'quadraticMineq');

% Defining the polynomial matrix F(x) of degree 0
degreeF = 0;
[prog,F] = sospolymatrixvar(prog,monomials(x,0:degreeF),
                                                   [m,hat_N]);

% Computation of the the matrix M appearing in Vdot
M = d_hat_Z * [Bstar Astar] * [F ; Hx*Q]+
                     [F ; Hx*Q]' * [Bstar Astar]' * d_hat_Z';

% Constraint on -M to be an SOS
prog = sosmatrixineq (prog, -M,'quadraticMineq');
```

SOSTOOLS script

```
% Call solver
solver_opt.solver = 'sedumi';
prog = sossolve(prog,solver_opt);

% Extracting the solutions
Qs = sosgetsol(prog,Q);
Fs = sosgetsol(prog,F);
Ms = sosgetsol(prog,M);

% The computed controller
Ks = Fs*inv(Qs);
u = Ks*hat_Z;

% The computed Lyapunov function
V= [x1, x2]*Qs*[x1; x2];
```

```
% Closed-loop system
fcl=[x1-2*x2^2+u; -x2-x1*x2-2*x2^3];

% Computation of Vdot along the
% closed-loop dynamics
Vdot = diff(V,x1)*fcl(1)
                  +diff(V,x2)*fcl(2);

% Finding the Square Matricial
% Representation of -Vdot
[QminusVdot,ZetaminusVdot]=
                   findsos(-Vdot);
```

* A. Papachristodoulou, J. Anderson, G. Valmorbida, S. Prajna, P. Seiler, and P. Parrilo, "Sostools – version 3.03 sum of squares optimization toolbox for Matlab." User's guide, 2018.

SOSTOOLS script

After one runs the script, it returns the controller

$$u = -3.59x_1$$

and the Lyapunov function

$$V(x) = \hat{Z}(x)^\top \begin{bmatrix} 0.2709 & 0 \\ 0 & 0.3417 \end{bmatrix} \hat{Z}(X)$$

The command

```
[QminusVdot,ZetaminusVdot]= findsos(-Vdot);
```

returns the Square Matrix Representation

$$-\dot{V}(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}^\top \begin{bmatrix} 1.4059 & 0 & 0 & 0.8836 \\ 0 & 0.6834 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.8836 & 0 & 0 & 1.3668 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_1x_2 \\ x_2^2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_2^2 \end{bmatrix}^\top \begin{bmatrix} 1.4059 & 0 & 0.8836 \\ 0 & 0.6834 & 0 \\ 0.8836 & 0 & 1.3668 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2^2 \end{bmatrix}$$

which shows $\dot{V}(x) < 0$ for all $x \neq 0$

Reminder – what we understood previously

Under

$$\begin{array}{ll} \text{Prior 1} & \dot{x} = f_\star(x) + g_\star(x)u = A_\star Z(x) + B_\star W(x)u \\ \text{Prior 2} & \{d(t_k)\}_{k=0}^{T-1} \in \left\{ \{d^k\}_{k=0}^{T-1} : \sum_{k=0}^{T-1} d^k {d^k}^\top \preceq \epsilon T I_n \right\} \end{array}$$

let $\hat{Z}(x), H(x)$ be matrices of functions such that

$$Z(x) = H(x)\hat{Z}(x) \quad \text{with} \quad \hat{Z}(x) = \left[x^\top \ldots\right]^\top$$

If $\forall x \neq 0$ there exists $Q \succ 0$, matrix $F(x)$ and $\mu(x) > 0$ such that

$$\begin{bmatrix} \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top Z_c \dfrac{\partial \hat{Z}(x)}{\partial x}^\top + (\star)^\top + \mu(x) \dfrac{\partial \hat{Z}(x)}{\partial x} L \dfrac{\partial \hat{Z}(x)}{\partial x}^\top & \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top \\ \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix} & -\mu(x)\Theta \end{bmatrix} \prec 0$$

then

$$\begin{array}{rl} u = F(x)Q^{-1}\hat{Z}(x) & \text{globally asymptotically stabilizer} \\ V(x) = \hat{Z}(x)^\top Q^{-1}\hat{Z}(x) & \text{Lyapunov function} \end{array}$$

Towards data-driven control design for polynomial systems

To make the previous condition tractable via SOS conditions, we focus on the polynomial case

- $Z(x), \hat{Z}(x), H(x)$ are all polynomial matrices satisfying

$$Z(x) = H(x)\hat{Z}(x)$$

- We seek a polynomial controller

$$u = K(x)\hat{Z}(x) \quad \text{with } K(x) \text{ polynomial matrix}$$

  and a polynomial Lyapunov function

$$V(x) = \hat{Z}(x)^\top P \hat{Z}(x) \quad \text{with } P \succ 0$$

- Properties of $\hat{Z}(x)$ (as before)

$$\begin{aligned}
&\hat{Z}(x) = 0 \Leftrightarrow x = 0 \ (\Rightarrow V(x) \text{ globally positive definite}) \\
&\hat{Z}(x) \text{ radially unbounded } (\Rightarrow V(x) \text{ radially unbounded}) \\
&\hat{Z}(x) \text{ contains the state vector } x
\end{aligned}$$

A polynomial matrix condition

Having imposed the decision variables $F(x), \mu(x)$ to be polynomials, the matrix appearing in the condition

$$-\begin{bmatrix} \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top Z_c \dfrac{\partial \hat{Z}(x)}{\partial x}^\top + (\star)^\top + \mu(x) \dfrac{\partial \hat{Z}(x)}{\partial x} L \dfrac{\partial \hat{Z}(x)}{\partial x}^\top & \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top \\ \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix} & -\mu(x)\Theta \end{bmatrix} \succ 0$$

becomes a polynomial matrix, and the condition itself becomes one of <u>certifying the strict positivity of a polynomial matrix</u>

From positivity to SOS conditions

Imposing the strict positivity of a polynomial (matrix) is NP-hard, hence we relax the requirement by imposing SOS conditions

Look for $Q \succ 0$, polynomial matrix $F(x)$ and polynomial $\mu(x) > 0$ such that

$$-\begin{bmatrix} \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top Z_c \dfrac{\partial \hat{Z}(x)}{\partial x}^\top + (\star)^\top + \mu(x)\dfrac{\partial \hat{Z}(x)}{\partial x} L \dfrac{\partial \hat{Z}(x)}{\partial x}^\top & \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top \\ \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix} & -\mu(x)\Theta \end{bmatrix}$$

is an SOS polynomial matrix

Rationale[*] "SOS polynomials that vanish at some points in space lie on the boundary of the cone of SOS polynomials, while interior point algorithms will look for the analytic center of the feasibility set, which is away from the boundary. Hence, SOS conditions will automatically aim at strict positivity if such solutions are feasible".

[*] A.A. Ahmadi. "Non-monotonic Lyapunov functions for stability of nonlinear and switched systems: theory and computation", Master's thesis, Massachusetts Institute of Technology, p. 41, 2008.

Data-driven control for polynomial systems

Theorem Consider a system

$$\dot{x} = A_\star Z(x) + B_\star W(x)u + d$$

with dataset $U_0, X_0, X_1, Z_0, \overline{U}_0$ satisfying $X_1 = A_\star Z_0 + B_\star \overline{U}_0 + D_0$.
Assume that

$$\overline{W}_0 = \begin{bmatrix} \overline{U}_0 \\ Z_0 \end{bmatrix} \quad \text{has full row rank}$$

and

$$D_0 \in \mathcal{D}_e = \{D \colon DD^\top \preceq R_D R_D^\top\}$$

If there exist $Q \succ 0$, polynomial matrix $F(x)$ and polynomial $\mu(x) > 0$ such that

$$-\begin{bmatrix} \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top Z_c \dfrac{\partial \hat{Z}(x)}{\partial x}^\top + (\star)^\top + \mu(x) \dfrac{\partial \hat{Z}(x)}{\partial x} L \dfrac{\partial \hat{Z}(x)}{\partial x}^\top & \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top \\ \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix} & -\mu(x)\Theta \end{bmatrix}$$

is an SOS polynomial matrix, then

$$\begin{aligned} u &= F(x)Q^{-1}\hat{Z}(x) && \text{globally asymptotically stabilizer} \\ V(x) &= \hat{Z}(x)^\top Q^{-1}\hat{Z}(x) && \text{Lyapunov function} \end{aligned}$$

Remarks I

- The approach is based on choosing the Lyapunov function $V(x) = \hat{Z}(x)^\top P \hat{Z}(x)$, and designing a controller that deals with the quadratic uncertainty induced by $DD^\top \preceq R_D R_D^\top$ and imposing SOS conditions on a matrix of polynomials.
- One could impose P to be polynomial as well, but this would complicate considerably the analysis
- Solution based on a data-dependent SOS program
- Other designer's choices: degrees of the controller gain $F(x)$ and of the polynomial multiplier $\mu(x)$
- The SOS program has size $(\hat{N} + N + M) \times (\hat{N} + N + M)$ with $\hat{N}, N$ size of vectors $\hat{Z}(x), Z(x)$, M number of rows of $W(x)$
- The decision variables are the $m \times \hat{N}$ polynomial matrix $F(x)$ and the $\hat{N} \times \hat{N}$ matrix P and the scalar multiplier $\mu(x)$. Thus, using $\hat{Z}(x)$ rather than $Z(x)$ helps lowering the number of decision variables

M. Guo, C. De Persis, P. Tesi. "Learning Control for Polynomial Systems Using Sum of Squares". CDC 2020.
M. Guo, C. De Persis, P. Tesi. "Data-driven stabilization of nonlinear polynomial systems with noisy data". IEEE Transactions on Automatic Control, provisionally accepted, ArXiv:2011.07833, 2020.

A simple example

Unknown system

$$\dot{x} = x^2 + u + d$$

Priors $f(x)$ polynomial of degree 2 and $g(x)$ polynomial of degree 0.
Hence

$$A_\star = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad B_\star = 1$$

$$Z(x) = \begin{bmatrix} x \\ x^2 \end{bmatrix} \quad W(x) = 1$$

Furthermore, we set

$$\hat{Z}(x) = x \quad \text{and} \quad H(x) = \begin{bmatrix} 1 \\ x \end{bmatrix}$$

The disturbance d that affects the system is such that $d(t) \in [-\epsilon, \epsilon]$ for all t, with $\epsilon = 0.05$.

An experiment is performed over the time interval $[0, 2.5]$, initializing the system in the interval $[-1, 1]$ and with an input sequence satisfying $u(t) \in [-1, 1]$ for all t.

Example

```
% Experiment carried out on the time interval [0,tt]
% value of the solutions are seeked at the k times
% specified in t1
tt=2.5;k=8*tt+1;
t1=linspace(0,tt,k)';

% Vector field f(x)+g(x)u= x^2 +u
n=1; m=1;

% Initial conditions

% infty-norm on x0
x0infty=1;

% randomly generate row vector of initial condition in the
% square of length x0infty
x0=(rand([1,n])-0.5.*ones(1,n))*x0infty;
u0=zeros(1,m);
```

Example

```
% Input in [-1,1] used during the experiment
uexp=(rand([length(t1),1])-0.5*ones([length(t1),1]))*2;

% Disturbance during the experiment
% Max Euclidean norm of disturbance

dmagn=0.1;

% Disturbance
dexp1=(rand([length(t1),1])-0.5*ones([length(t1),1]))*dmagn;

% Execution of the experiment
[t,y]=ode45(@(t,y) simplepoly(t,y,t1,uexp,dexp1),t1,x0);
```

Example

```
% Data collection
% T <= k samples are selected for control design from data
T=k-1;

% Z(x) as in the factorization f(x)= AZ(x);
% for f(x) as above Z(x) = [x ; x^2], W(x)=1
N=2; M=1;

% Initialization of the matrix of data
U0=zeros(m,T); X0=zeros(n,T); X1=zeros(n,T); Z0=zeros(N,T);

for i=1:T
    U0(1,i)=uexp(i); D0(1,i)=dexp1(i); X0(:,i)=y(i,:)';
    u(i)=uexp(i); d(i,1)=dexp1(i);
    % Noisy derivative measurements x_dot = f(x)+g(x)u=[x^2+u]
    X1(:,i)= [y(i,1)^2+u(i)+d(i,1)];
    % Z(x) = [x ; x^2]
    Z0(:,i)= [y(i,1); y(i,1)^2];
end
```

Example

```
% To carry out the design, W0 must be full row rank
W0=[U0 ; Z0];

% Noise free least square estimate of [B A]'
ZcTranspose=X1*pinv(W0);

% Other matrices of data used in the SOS program
L = X1*pinv(W0)*W0*X1'-(X1*X1'-dmagn*T*eye(n));
% Here U stands for Theta in the slides
U = W0*W0';

% Exemplifying an element of the uncertainty set \mathcal{C}

InvSqrtU=inv(sqrtm(U)); sqrtL=sqrtm(L);
SizeInvSqrtU=size(InvSqrtU); SizeSqrtL=size(sqrtL);
Yaux=rand(SizeInvSqrtU(1),SizeSqrtL(2)); Y=Yaux/(norm(Yaux));

% An element of the feasible set of matrices under noisy measurements
BAestimate=ZcTranspose-sqrtm(L)*Y'*inv(sqrtm(U))
```

An element of $\mathcal{C}$ $\left[\begin{array}{c|c} B & A \end{array}\right] = \left[\begin{array}{c|cc} 1.1366 & 2.8327 & 6.6046 \end{array}\right]$

Example

At this stage we have all the elements to write down the matrix $M(x)$

$$-\begin{bmatrix} \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top Z_c \dfrac{\partial \hat{Z}(x)}{\partial x}^\top + (\star)^\top + \mu(x) \dfrac{\partial \hat{Z}(x)}{\partial x} L \dfrac{\partial \hat{Z}(x)}{\partial x}^\top & \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix}^\top \\ \begin{bmatrix} W(x)F(x) \\ H(x)Q \end{bmatrix} & -\mu(x)\Theta \end{bmatrix} =$$

$$-\begin{bmatrix} \begin{bmatrix} F(x) \\ \begin{bmatrix} 1 \\ x \end{bmatrix} Q \end{bmatrix}^\top Z_c + (\star)^\top + \mu(x) L & \begin{bmatrix} F(x) \\ \begin{bmatrix} 1 \\ x \end{bmatrix} Q \end{bmatrix}^\top \\ \begin{bmatrix} F(x) \\ \begin{bmatrix} 1 \\ x \end{bmatrix} Q \end{bmatrix} & -\mu(x)\Theta \end{bmatrix}$$

To make $M(x)$ of even degree we need to require $F(x)$ and/or $\mu(x)$ of even degree. It turns out that having both of degree 2 is numerically better. Hence

$$\begin{bmatrix} F(x) \\ \begin{bmatrix} 1 \\ x \end{bmatrix} Q \end{bmatrix} = \begin{bmatrix} c_2 + c_3 x + c_4 x^2 \\ c_1 \\ c_1 x \end{bmatrix} \text{ with } \begin{array}{l} Q = c_1 \;\; F(x) = c_2 + c_3 x + c_4 x^2 \\ \mu(x) = c_7 + c_6 x + c_5 x^2 \end{array}$$

Example

As in the previous example, we can code SOSTOOLS to look for $Q \succ 0, F(x)$ and SOS polynomial $\mu(x)$ (i.e., $c_1, \ldots, c_7$) that render $M(x)$ an SOS polynomial matrix, if feasible.

In this case, it returns

$$\begin{aligned} V(x) &= \quad 0.052x^2 \\ F(x) &= \quad -16.94x^2 - 2.52x - 17.49 \\ \mu(x) &= \quad 0.391x^2 + 0.0155x + 0.3951 \end{aligned}$$

with

$$\dot{V}(x) = -\begin{bmatrix} x \\ x^2 \end{bmatrix}^\top \begin{bmatrix} 1.8116 & 0.0788 \\ 0.0788 & 1.7552 \end{bmatrix} \begin{bmatrix} x \\ x^2 \end{bmatrix}$$

Remarks

- The uncertainty introduced by the noise can give raise to systems that are dramatically different from the ground-truth one (i.e. dense $A_\star$, $B_\star$, higher degree polynomials, etc)
- Inclusion of as many priors as possible and increased quality of data are important factors to successfully carry out control synthesis
- Priors such as so-called "side information" (invariance of sets, decrease of energy functions, etc.) can be included in the SOS program as constraints[⋆]
- Disturbances that fulfil instantaneous bounds (Lecture 3) rather than energy like bounds might lead to smaller uncertainty sets whose size decreases with larger data sets
- The size of SOS programs grows polynomially with the dimension of the state and exponentially with the degree of the polynomials involved can be tackled via scalable alternatives to SOS synthesis[∘]

[⋆] A.A. Ahmadi, B. El Khadir. "Learning Dynamical Systems with Side Information". arXiv:2008.10135, 2020.
[∘] A.A. Ahmadi, G. Hall, A. Papachristodoulou, J. Saunderson, Y. Zheng. "Improving efficiency and scalability of sum of squares optimization: Recent advances and limitations". Proceedings 56th CDC, 453–462, 2017.

Other topics...

....for which there was no time

- The case with noisy measurements
- Local asymptotic stabilizers
- Bilinear systems (Exercise #1)

Selected references

This lecture mostly follows:

M. Guo, C. De Persis, P. Tesi. Data-driven stabilization of nonlinear polynomial systems with noisy data. IEEE Transactions on Automatic Control 2022.

Other results (e.g., local stabilization, instantaneous bounds, etc.):

A. Bisoffi, C. De Persis, P. Tesi. Data-driven control via Petersen's lemma. Automatica 2022

Design of control invariant sets (safe control) is discussed in:

A. Luppi, A. Bisoffi, C. De Persis, P. Tesi. Data-driven design of safe control for polynomial systems. arXiv:2112.12664, 2021.

The design of input-to-state stabilising controllers is studied in:

H. Chen, A. Bisoffi, C. De Persis. Learning input-to-state stability with respect to measurement error from data. IEEE Conference on Decision and Control 2023

Control design for general nonlinear systems via Taylor's expansion is considered in:

M. Guo, A. Bisoffi, C. De Persis. Data-driven stabilizer design and closed-loop analysis of general nonlinear systems via Taylor's expansion. ArXiv:2209.01071 2021

Conclusions

The lectures have covered methods for the “direct” design of data-driven control policies

- The methods lead to data-dependent formulas based on LMI, SDP, SOS
- Data collected in a low-complexity one-shot experiment
- Robustness to noise
- Noise modeled as a bounded energy signal or fulfilling instantaneous bounds
- Good potentials for nonlinear systems
- Much of the work is in progress
- The proposed methods have many connections with other existing approaches in learning for control, all to be explored
- Exciting opportunities

Outlook

- LMIs, SDP are ubiquitous in control – this approach can be used to deal with manifold problems replacing models with data.
- Data-driven SOS can be extensively developed for tackling a variety of control tasks along with techniques that limit their computational complexity.
- Nonlinear dynamics Use approximate data-dependent representations of nonlinear systems to learn control policies[•]
- Exploiting structure of nonlinear systems for less conservative design[⋆]
- Dynamic output feedback for nonlinear systems[∘]
- Safe control Include state and input constraint in the data-based control design to obtain explicit formulas for safe controllers.
- Design of (control) Lyapunov functions from data.
- "Real-life" applications to show the effectiveness of these methods for challenging engineering problems.

[•] C. De Persis, M. Rotulo, P. Tesi. Learning controllers from data via approximate nonlinearity cancellation. IEEE Transactions on Automatic Control 2023.

[⋆] M. Guo, C. De Persis, P. Tesi. Learning control of second-order systems via nonlinearity cancellation. IEEE Conference on Decision and Control 2023.

[∘] X. Dai, C. De Persis, N. Monshizadeh, P. Tesi. Data-driven control of nonlinear systems from input-output data. IEEE Conference on Decision and Control 2023.

Appendix

Exercises

Exercise#1 *The aim of this exercise is to show how Petersen's lemma-based arguments can be also effectively used to design model-based controllers for bilinear system. The arguments can be extended to deal with the data-driven case and have the advantage to require the solution of an LMI rather than an SOS program.*

Consider the single-input bilinear system

$$\dot{x} = Ax + Bu + uDx$$

with $x \in \mathbb{R}^n, u \in \mathbb{R}$.
(a) Let $V(x) = x^\top Px$, with $P \succ 0$, and $u = Kx$. Determine the state-dependent matrix $\mathcal{P}(x)$ in the expression of the Lyapunov inequality

$$\dot{V}(x) = x^\top P\,\mathcal{P}(x)\,Px$$

Answer. Straightforward calculations show that

$$\mathcal{P}(x) = (A+BK)P^{-1} + P^{-1}(A+BK)^\top + DxKP^{-1} + P^{-1}K^\top x^\top D^\top$$

Exercises

(b) Show whether or not the condition

$$\exists P \succ 0, K \,\text{such that}\, \hat{\mathcal{P}}(\delta) \prec 0 \;\forall \delta \in \mathbb{R}^n \text{ such that } \delta^\top \delta \leq 1$$

where

$$\hat{\mathcal{P}}(\delta) = (A + BK)P^{-1} + P^{-1}(A + BK)^\top + DP^{-1/2}\delta K P^{-1} + P^{-1}K^\top \delta^\top P^{-1/2} D^\top$$

implies the condition

$$\exists P \succ 0, K \,\text{such that}\, \mathcal{P}(x) \prec 0 \;\forall x \in \mathbb{R}^n \text{ such that } x^\top P x \leq 1$$

Answer. Consider the matrix $\mathcal{P}(x)$ for any x such that $x^\top P x \leq 1$. Define $\delta = P^{1/2}x$, which is well defined because $P \succ 0$. Then $\mathcal{P}(x)|_{x=P^{-1/2}\delta} = \hat{\mathcal{P}}(\delta)$. Moreover, $\delta^\top \delta = x^\top P^{1/2} P^{1/2} x = x^\top P x \leq 1$. Hence, $\mathcal{P}(x)|_{x=P^{-1/2}\delta} = \hat{\mathcal{P}}(\delta) \prec 0$ and by the genericity of x, the implication holds.

(c) Using Petersen's lemma, show that $\dot{V}(x) < 0$ for all $x \neq 0$ such that $x^\top P x \leq 1$ if there exist $Q \succ 0$, $Q \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{1\times n}$ and $\epsilon > 0$ such that the following matrix inequality is satisfied

$$\begin{bmatrix} \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} F \\ Q \end{bmatrix} + \begin{bmatrix} F \\ Q \end{bmatrix}^\top \begin{bmatrix} B & A \end{bmatrix}^\top + \epsilon D Q D^\top & F^\top \\ F & -\epsilon \end{bmatrix} \prec 0$$

Exercises

Answer. We use Petersen's lemma for $\hat{\mathcal{P}}(\delta)$. Observe that

$$\hat{\mathcal{P}}(\delta) = \underbrace{(A+BK)P^{-1} + P^{-1}(A+BK)^\top}_{\mathcal{G}} + \underbrace{DP^{-1/2}}_{\mathcal{M}}\,\delta\,\underbrace{KP^{-1}}_{\mathcal{N}} + P^{-1}K^\top\delta^\top P^{-1/2}D^\top$$

and that $\hat{\mathcal{P}}(\delta) \prec 0$ for all $\delta \in \mathbb{R}^n$ such that $\delta^\top\delta \leq 1$ if and only there exists $\epsilon > 0$ such that

$$\begin{aligned} 0 \succ\;\; & \mathcal{G} + \epsilon\mathcal{M}\mathcal{M}^\top + \epsilon^{-1}\mathcal{N}^\top\mathcal{N} \\ = \;\; & (A+BK)P^{-1} + P^{-1}(A+BK)^\top + \epsilon DP^{-1}D^\top + \epsilon^{-1}(KP^{-1})^\top KP^{-1} \end{aligned}$$

Changing the variables as

$$F := KP^{-1} \quad Q := P^{-1}$$

one obtains the equivalent inequality

$$\begin{bmatrix} B & A \end{bmatrix}\begin{bmatrix} F \\ Q \end{bmatrix} + \begin{bmatrix} F \\ Q \end{bmatrix}^\top \begin{bmatrix} B & A \end{bmatrix}^\top + \epsilon DQD^\top + F^\top + \epsilon^{-1}F^\top F \prec 0$$

An application of Schur complement returns the solution.

Exercises

Exercise #2 *The purpose of this exercise is to show that, if at each sampling time the disturbance satisfies linear constraints (instead of the quadratic ones adopted during the lectures), then the set of feasible system matrices is a bounded polyhedral set if and only if the data satisfy a persistence of excitation condition.*

Consider the linearly parametrized nonlinear system

$$\dot{x} = A_\star Z(x) + B_\star W(x)u + d$$

with $x \in \mathbb{R}^n, u \in \mathbb{R}^m, Z(x) \in \mathbb{R}^N, W(x) \in \mathbb{R}^M$, and the dataset

$$\left\{u(t_i), x(t_i), \dot{x}(t_i), \overline{z}(t_i), \overline{u}(t_i),\ i = 0, 1, \ldots, T-1\right\}$$

satisfying

$$\dot{x}(t_i) = A_\star \overline{z}(t_i) + B_\star \overline{u}(t_i) + d(t_i) \quad i = 0, 1, \ldots, T-1$$

where

$$\begin{aligned} \overline{z}(t_i) :=& \quad Z(x(t_i)) \\ \overline{u}(t_i) :=& \quad W(x(t_i))u(t_i). \end{aligned}$$

For each $i = 0, \ldots, T-1$, let the disturbance vector $d(t_i)$, with $i = 0, \ldots, T-1$, belong to the set

$$\mathcal{D}_i := \left\{d \in \mathbb{R}^n :\ -\epsilon \mathbb{1}_n \le d \le \epsilon \mathbb{1}_n\right\}$$

for some known $\epsilon > 0$, where the inequalities in the definition of $\mathcal{D}_i$ must be understood componentwise and $\mathbb{1}_n$ is the n-dimensional vector of all ones. Consider the feasible set of system's matrices

$$\mathcal{I} := \left\{(A, B) :\ -\epsilon \mathbb{1}_n \le \dot{x}(t_i) - \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} \overline{u}(t_i) \\ \overline{z}(t_i) \end{bmatrix} \le \epsilon \mathbb{1}_n, \quad \forall i = 0, 1, \ldots, T-1\right\}$$

Exercises

(a) Use the identity $\mathrm{vec}(AXB) = (B^\top \otimes A)\mathrm{vec}(X)$ [1] to show that

$$\mathcal{I} = \left\{ (A, B) \colon \; -\epsilon \mathbb{1}_{nT} - \mathrm{vec}(X_1) \leq -\left(\begin{bmatrix} \overline{U}_0 \\ Z_0 \end{bmatrix}^\top \otimes I_n \right) \mathrm{vec}\left(\begin{bmatrix} B & A \end{bmatrix} \right) \leq \epsilon \mathbb{1}_{nT} - \mathrm{vec}(X_1) \right\}$$

where

$$\mathrm{vec}(X_1) = \begin{bmatrix} \dot{x}(t_0) \\ \dot{x}(t_1) \\ \vdots \\ \dot{x}(t_{T-1}) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \overline{U}_0 \\ Z_0 \end{bmatrix} := \begin{bmatrix} \overline{u}(t_0) & \overline{u}(t_1) & \dots & \overline{u}(t_{T-1}) \\ \overline{z}(t_0) & \overline{z}(t_1) & \dots & \overline{z}(t_{T-1}) \end{bmatrix}$$

[1] The symbol $\otimes$ denotes the Kronecker product and $\mathrm{vec}(Y)$ is the vectorization of matrix $Y = \begin{bmatrix} Y_1 & Y_2 & \dots & Y_q \end{bmatrix} \in \mathbb{R}^{p \times q}$, i.e. $\mathrm{vec}(Y) = \begin{bmatrix} Y_1 \\ \dots \\ Y_q \end{bmatrix}$.

Exercises

Answer. Observe that

$$\begin{aligned}\dot{x}(t_i) - \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} \overline{u}(t_i) \\ \overline{z}(t_i) \end{bmatrix} &= \dot{x}(t_i) - \left(\begin{bmatrix} \overline{u}(t_i) \\ \overline{z}(t_i) \end{bmatrix}^\top \otimes I_n \right) \operatorname{vec}(\begin{bmatrix} B & A \end{bmatrix}) \\ &= \dot{x}(t_i) - \left(\begin{bmatrix} \overline{u}(t_i)^\top & \overline{z}(t_i)^\top \end{bmatrix} \otimes I_n \right) \operatorname{vec}(\begin{bmatrix} B & A \end{bmatrix})\end{aligned}$$

Hence, $(A, B) \in \mathcal{I}$ *if and only if* $\operatorname{vec}(\begin{bmatrix} B & A \end{bmatrix})$ *satisfies*

$$\begin{aligned}-\epsilon \begin{bmatrix} \mathbb{1}_n \\ \mathbb{1}_n \\ \vdots \\ \mathbb{1}_n \end{bmatrix} - \begin{bmatrix} \dot{x}(t_0) \\ \dot{x}(t_1) \\ \vdots \\ \dot{x}(t_{T-1}) \end{bmatrix} \leq -\left(\begin{bmatrix} \overline{u}(t_0)^\top & \overline{z}(t_0)^\top \\ \overline{u}(t_1)^\top & \overline{z}(t_1)^\top \\ \vdots & \\ \overline{u}(t_{T-1})^\top & \overline{z}(t_{T-1})^\top \end{bmatrix} \otimes I_n \right) \operatorname{vec}(\begin{bmatrix} B & A \end{bmatrix}) \\ \leq \epsilon \begin{bmatrix} \mathbb{1}_n \\ \mathbb{1}_n \\ \vdots \\ \mathbb{1}_n \end{bmatrix} - \begin{bmatrix} \dot{x}(t_0) \\ \dot{x}(t_1) \\ \vdots \\ \dot{x}(t_{T-1}) \end{bmatrix}.\end{aligned}$$

Exercises

(b) Use the following result

<u>Lemma</u> (Blanchini-Miani)*Let $\mathcal{A} = \{x \colon b_1 \leq Ax \leq b_2\}$ be a nonempty set with $A \in \mathbb{R}^{m\times n}, b_1, b_2 \in \mathbb{R}^m$. Then $\mathcal{A}$ is bounded if and only if A has full column rank.*

to show that $\mathcal{I}$ is bounded if and only if $\begin{bmatrix} \overline{U}_0 \\ Z_0 \end{bmatrix}$ has full row rank.

Answer. First we observe that the set $\mathcal{I}$ is nonempty because $(A_\star, B_\star)$ belongs to it. To continue, recall the following property of the Kronecker product: $\operatorname{rank}(A \otimes B) = \operatorname{rank}(A)\operatorname{rank}(B)$. Hence,

$$\operatorname{rank}\left(\begin{bmatrix} \overline{U}_0 \\ Z_0 \end{bmatrix}^\top \otimes I_n\right) = \operatorname{rank}\left(\begin{bmatrix} \overline{U}_0 \\ Z_0 \end{bmatrix}^\top\right) n$$

We conclude that the $nT \times n(N+M)$ matrix $\begin{bmatrix} \overline{U}_0 \\ Z_0 \end{bmatrix}^\top \otimes I_n$ has full column rank if and only if

$\operatorname{rank}\left(\begin{bmatrix} \overline{U}_0 \\ Z_0 \end{bmatrix}^\top\right) = N + M$, i.e. if and only if $\begin{bmatrix} \overline{U}_0 \\ Z_0 \end{bmatrix}$ has full row rank.

Exercises

Exercise #3 *The purpose of this exercise is to provide a different condition for the design of a stabilizing controller for polynomial systems in the case a control Lyapunov function is available.*

Consider the system $\dot{x} = A_\star Z(x) + B_\star W(x)u$, with $x \in \mathbb{R}^n, u \in \mathbb{R}$, and $Z(x) \in \mathbb{R}^N, W(x) \in \mathbb{R}^M$ known vectors of monomials. Assume that a continuously differentiable, radially unbounded and globally positive definite polynomial function $V \colon \mathbb{R}^n \to \mathbb{R}$ is known for which there exists a polynomial function $k(x)$ such that $\frac{\partial V}{\partial x}^\top (A_\star Z(x) + B_\star W(x)k(x)) < 0$ for all $x \neq 0$.
The pair $(B_\star, A_\star)$ is known to belong to the set

$$\mathcal{I} = \left\{ (A, B) \colon \ \mathsf{N}\mathrm{vec}(\begin{bmatrix} B & A \end{bmatrix}) \leq \mathsf{e} \right\}$$

where the inequality holds componentwise and $\mathsf{N} \in \mathbb{R}^{2nT \times n(M+N)}, \mathsf{e} \in \mathbb{R}^{2nT}$ are known matrices of data (see Exercise 2(a) for an expression of these matrices).

Exercises

(a) Find a vector $\mathsf{d}(x) \in \mathbb{R}^{n(M+N)}$ such that

$$\frac{\partial V}{\partial x}(AZ(x) + BW(x)k(x)) = \mathsf{d}(x)^\top \mathrm{vec}(\begin{bmatrix} B & A \end{bmatrix})$$

for any $(A, B) \in \mathcal{I}$.

Answer. We use the identity $\mathrm{vec}(AXB) = (B^\top \otimes A)\mathrm{vec}(X)$ once again on

$$\frac{\partial V}{\partial x}(AZ(x) + BW(x)k(x)) = \frac{\partial V}{\partial x}\begin{bmatrix} B & A \end{bmatrix}\begin{bmatrix} W(x)k(x) \\ Z(x) \end{bmatrix}$$

to obtain

$$\left(\begin{bmatrix} W(x)k(x) \\ Z(x) \end{bmatrix}^\top \otimes \frac{\partial V}{\partial x}\right)\mathrm{vec}(\begin{bmatrix} B & A \end{bmatrix}) = \mathsf{d}(x)^\top \mathrm{vec}(\begin{bmatrix} B & A \end{bmatrix})$$

where

$$\mathsf{d}(x) := \begin{bmatrix} W(x)k(x) \\ Z(x) \end{bmatrix} \otimes \frac{\partial V}{\partial x}^\top.$$

Exercises

(b) Use the following version of Farkas' lemma

Lemma (Mangasarian, Dai-Sznaier) *Consider matrices* $N \in \mathbb{R}^{\mu \times \nu}$, $d \in \mathbb{R}^{\nu}$, $e \in \mathbb{R}^{\mu}$. *Assume* $Nz \leq e$ *is feasible. Then, the inclusion*

$$\left\{z\colon\ Nz \leq e\right\} \subseteq \left\{z\colon\ d^\top z < 0\right\}$$

holds if and only if

$$\exists w \in \mathbb{R}^{\mu}\colon N^\top w = d, \quad e^\top w < 0, \quad w \geq 0$$

to show that for any $x \neq 0$

$$\frac{\partial V}{\partial x}(AZ(x) + BW(x)k(x)) < 0 \ \forall (A, B) \in \mathcal{I}$$

if and only if

$$\exists y(x) \in \mathbb{R}^{2nT}\colon \mathsf{N}^\top y(x) = \mathsf{d}(x), \quad \mathsf{e}^\top y(x) < 0, \quad y(x) \geq 0$$

Exercises

Answer. Note that

$$\frac{\partial V}{\partial x}(AZ(x) + BW(x)k(x)) < 0 \ \forall (A,B) \in \mathcal{I}$$

is equivalently stated as the inclusion

$$\Big\{(A,B)\colon \ \mathsf{N}\mathrm{vec}(\begin{bmatrix} B & A\end{bmatrix}) \leq \mathsf{e}\Big\} \subseteq \Big\{(A,B)\colon \ \mathsf{d}(x)^\top \mathrm{vec}(\begin{bmatrix} B & A\end{bmatrix}) < 0\Big\}$$

By pointwise application of Farkas' lemma, the inclusion holds if and only if

$$\exists y(x) \in \mathbb{R}^{2nT} : \mathsf{N}^\top y(x) = \mathsf{d}(x) \quad \mathsf{e}^\top y(x) < 0 \quad y(x) \geq 0$$

Exercises

(c) Relax the previous condition to a sufficient SOS-based condition in the polynomial decision variables $y(x), k(x)$.

Answer. The relaxation takes the form

$$\exists y(x) \in \Sigma^{2nT}, k(x) \in \mathcal{P} \colon \mathsf{N}^\top y(x) = \begin{bmatrix} \left(W(x) \otimes \frac{\partial V}{\partial x}^\top\right) k(x) \\ Z(x) \otimes \frac{\partial V}{\partial x}^\top \end{bmatrix}, \quad -\mathsf{e}^\top y(x) \in \Sigma$$

where Σ^q is the set of q-dimensional SOS polynomial vectors (with $\Sigma^1 = \Sigma$ for the sake of simplicity) and $\mathcal{P}$ is the set of polynomials.