Lecture 1: Market clearing as an optimization problem

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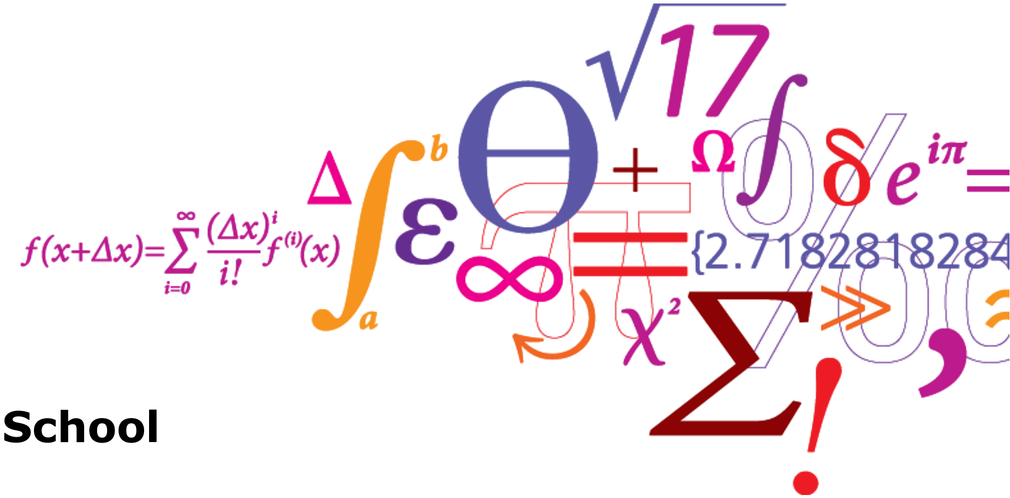

The 6th KIOS Graduate Training School

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Market clearing: a simple example

Let us get started with a question. Assume an electricity market with a single generator (G1) and an elastic demand (D1). What are the market-clearing outcomes (production, consumption and market-clearing price)?

Capacity: 100 MW
Offer price: $12/MWh

Demand D1

Maximum load: 80 MW
Bid price: $40/MWh

Market outcomes:

- Production level of G1: **?**
- Consumption level of D1: **?**
- Market-clearing price: **?**

Market clearing: a simple example

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Offer price: $12/MWh

Demand D1

Maximum load: 80 MW
Bid price: $40/MWh

Market outcomes:

- Production level of G1: **80 MW**
- Consumption level of D1: **80 MW**
- Market-clearing price: **$12/MWh**

Market clearing: a simple example

An extended example: two generators (G1 and G2) and two elastic demands (D1 and D2)

Capacity: 100 MW
Offer price: $12/MWh

Capacity: 80 MW
Offer price: $20/MWh

Demand D1

Maximum load: 100 MW
Bid price: $40/MWh

Demand D2

Maximum load: 50 MW
Bid price: $35/MWh

Market outcomes:

- Productions of G1 and G2: **?**
- Consumptions of D1 and D2: **?**
- Market-clearing price: **?**

Market clearing: a simple example

An extended example: two generators (G1 and G2) and two elastic demands (D1 and D2)

Capacity: 100 MW
Offer price: $12/MWh

Capacity: 80 MW
Offer price: $20/MWh

Demand D1

Maximum load: 100 MW
Bid price: $40/MWh

Demand D2

Maximum load: 50 MW
Bid price: $35/MWh

Market outcomes:

- Productions of G1 and G2: **100 MW and 50 MW**
- Consumptions of D1 and D2: **100 MW and 50 MW**
- Market-clearing price: **$20/MWh**

Market clearing as an optimization problem

Question:
How to form the previous example as an optimization problem?

Market clearing as an optimization problem

Generic form:

Maximize social welfare (SW) of the market[1]

Subject to

- All technical constraints of generators and demands
- Power balance equality

[1] SW (also known as "*market surplus*") is equal to:
[total utility of demands based on their bid prices] – [total cost of generators based on their offer prices]

Market clearing as an optimization problem

$$\underset{p^{\mathrm{G1}},p^{\mathrm{G2}},p^{\mathrm{D1}},p^{\mathrm{D2}}}{\text{Maximize}} \quad SW = [40p^{\mathrm{D1}} + 35p^{\mathrm{D2}}] - [12p^{\mathrm{G1}} + 20p^{\mathrm{G2}}] \tag{1a}$$

subject to:

$$0 \leq p^{\mathrm{D1}} \leq 100 \tag{1b}$$

$$0 \leq p^{\mathrm{D2}} \leq 50 \tag{1c}$$

$$0 \leq p^{\mathrm{G1}} \leq 100 \tag{1d}$$

$$0 \leq p^{\mathrm{G2}} \leq 80 \tag{1e}$$

$$p^{\mathrm{D1}} + p^{\mathrm{D2}} - p^{\mathrm{G1}} - p^{\mathrm{G2}} = 0 \tag{1f}$$

Market clearing as an optimization problem

$$\underset{p^{G1},p^{G2},p^{D1},p^{D2}}{\text{Maximize}} \quad SW = \underbrace{[40p^{D1} + 35p^{D2}]}_{\text{Utility of demands}} - \underbrace{[12p^{G1} + 20p^{G2}]}_{\text{Cost of generators}} \tag{1a}$$

Set of primal variables

subject to:

$$0 \le p^{D1} \le 100 \tag{1b}$$

$$0 \le p^{D2} \le 50 \tag{1c}$$

Consumption limits (1b)–(1c)

$$0 \le p^{G1} \le 100 \tag{1d}$$

$$0 \le p^{G2} \le 80 \tag{1e}$$

Generation limits (1d)–(1e)

$$p^{D1} + p^{D2} - p^{G1} - p^{G2} = 0 \tag{1f}$$

Power balance (1f)

Market clearing as an optimization problem

DTU

$$\underset{p^{G1},p^{G2},p^{D1},p^{D2}}{\text{Maximize}} \quad SW = \underbrace{[40p^{D1} + 35p^{D2}]}_{\text{Utility of demands}} - \underbrace{[12p^{G1} + 20p^{G2}]}_{\text{Cost of generators}} \quad (1a)$$

Set of primal variables

subject to:

$$0 \leq p^{D1} \leq 100 \quad (1b)$$
$$0 \leq p^{D2} \leq 50 \quad (1c)$$

Consumption limits

$$0 \leq p^{G1} \leq 100 \quad (1d)$$
$$0 \leq p^{G2} \leq 80 \quad (1e)$$

Generation limits

$$p^{D1} + p^{D2} - p^{G1} - p^{G2} = 0 \quad (1f)$$

Power balance

Discussion:
Is this optimization problem convex? How to know it?

Market clearing as an optimization problem

Question:

How to obtain market-clearing price within the optimization problem?

Market clearing as an optimization problem

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Answer:
The dual variable (also known as "Lagrangian multiplier") of the power balance equality provides the market-clearing price!

Market clearing as an optimization problem

Question:

How to obtain market-clearing price within the optimization problem?

Answer:

The dual variable (also known as "Lagrangian multiplier") of the power balance equality provides the market-clearing price!

Note: This is based on "uniform" pricing scheme, which is the most common practice in real-world electricity markets. There are other types of pricing schemes, such as "pay-as-bid" and "Vickrey–Clarke–Groves (VCG)", which derive market prices in a different way.

Market clearing as an optimization problem

$$\underset{p^{G1},p^{G2},p^{D1},p^{D2}}{\text{Maximize}} \quad SW = [40p^{D1} + 35p^{D2}] - [12p^{G1} + 20p^{G2}] \tag{1a}$$

subject to:

$$0 \leq p^{D1} \leq 100 \tag{1b}$$

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$$0 \leq p^{G1} \leq 100 \tag{1d}$$

$$0 \leq p^{G2} \leq 80 \tag{1e}$$

$$p^{D1} + p^{D2} - p^{G1} - p^{G2} = 0 \qquad : \lambda \tag{1f}$$

Market clearing as an optimization problem

$$\underset{p^{G1},p^{G2},p^{D1},p^{D2}}{\text{Maximize}} \quad SW = [40p^{D1} + 35p^{D2}] - [12p^{G1} + 20p^{G2}] \tag{1a}$$

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$$p^{D1} + p^{D2} - p^{G1} - p^{G2} = 0 \quad : \lambda \tag{1f}$$

Dual variable of power balance equality

Market clearing as an optimization problem

$$\underset{p^{G1},p^{G2},p^{D1},p^{D2}}{\text{Maximize}} \quad SW = [40p^{D1} + 35p^{D2}] - [12p^{G1} + 20p^{G2}] \tag{1a}$$

subject to:

$$0 \leq p^{D1} \leq 100 \tag{1b}$$

$$0 \leq p^{D2} \leq 50 \tag{1c}$$

$$0 \leq p^{G1} \leq 100 \tag{1d}$$

$$0 \leq p^{G2} \leq 80 \tag{1e}$$

$$p^{D1} + p^{D2} - p^{G1} - p^{G2} = 0 \quad : \lambda \tag{1f}$$

Discussion:

- What does a dual variable show in general (mathematical interpretation)?
- What is its sign (negative, or positive, or free)? Can the electricity market price be negative?

Market clearing as an optimization problem

Compact form:

Market clearing as an optimization problem

Compact form:

$$\underset{p_g^{\mathrm{G}}, p_d^{\mathrm{D}}}{\text{Maximize}} \quad SW = \sum_d U_d \, p_d^{\mathrm{D}} - \sum_g C_g \, p_g^{\mathrm{G}} \qquad (1a)$$

subject to:

$$0 \leq p_d^{\mathrm{D}} \leq \overline{P}_d^{\mathrm{D}} \quad \forall d \qquad (1b)$$

$$0 \leq p_g^{\mathrm{G}} \leq \overline{P}_g^{\mathrm{G}} \quad \forall g \qquad (1c)$$

$$\sum_d p_d^{\mathrm{D}} - \sum_g p_g^{\mathrm{G}} = 0 \qquad : \lambda \qquad (1d)$$

U_d : bid price of demand d

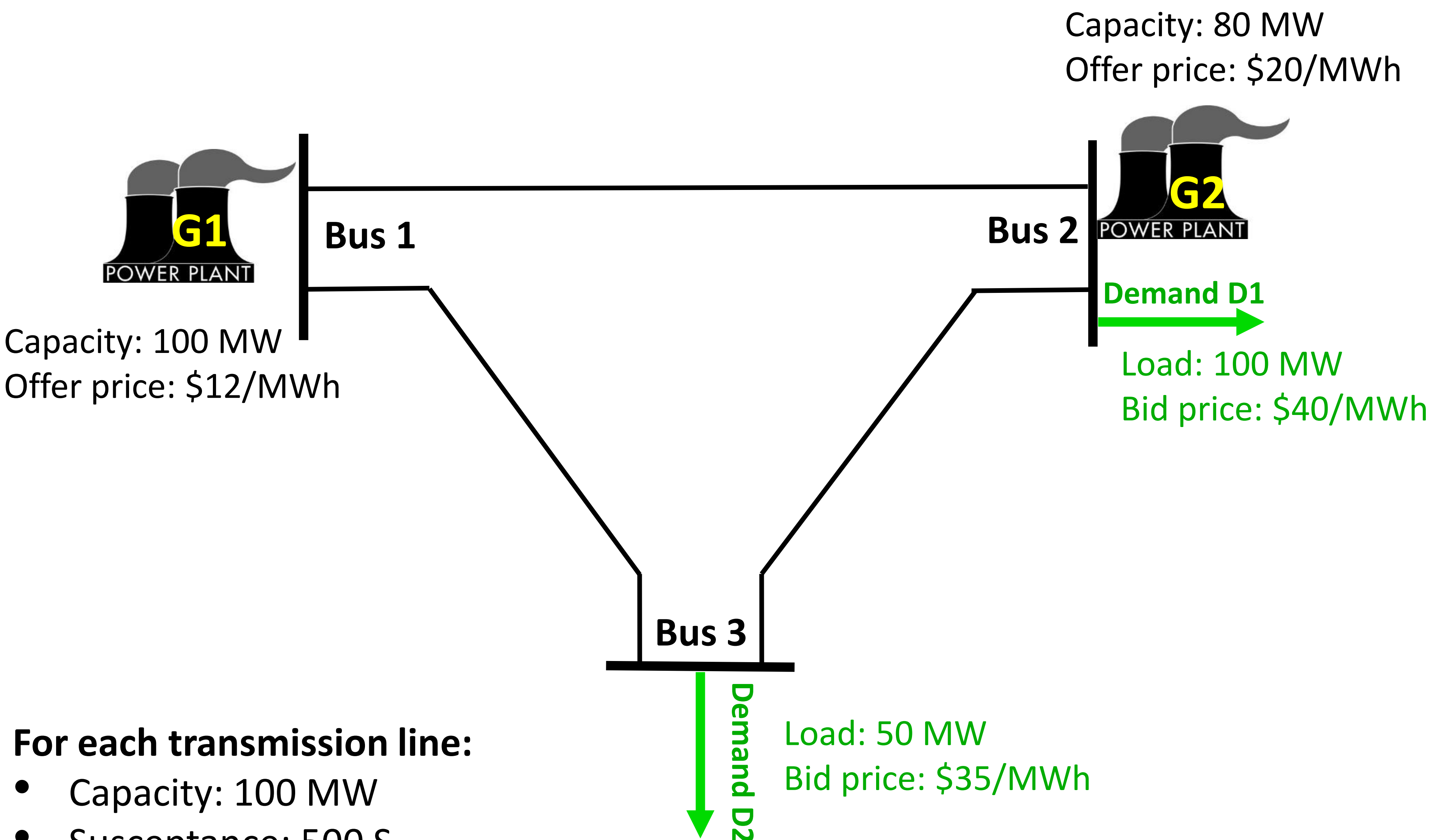
C_g : offer price of generator g

$\overline{P}_d^{\mathrm{D}}$: maximum load of demand d

$\overline{P}_g^{\mathrm{G}}$: capacity of generator g

Market clearing considering network

Market clearing considering network



For each transmission line:

- Capacity: 100 MW
- Susceptance: 500 S

Market clearing considering network

Let's use an approximate linearized representation of power flow equations (DC power flow). Accordingly, the following equation gives the power flow across the line connecting bus *n* to bus *m*:

$$f_{n,m} = B_{n,m}(\theta_n - \theta_m)$$

Power flow from bus *n* to bus *m* (variable) — $f_{n,m}$

Susceptance (parameter) — $B_{n,m}$

Difference of voltage angles of buses *n* and *m* (variable) — $(\theta_n - \theta_m)$

Market clearing considering network

Discussion:
- What are the power flow equations in reality?
- Why do we use the DC power flow equations?

$$f_{n,m} = B_{n,m}(\theta_n - \theta_m)$$

Power flow from bus *n* to bus *m* (variable)

Susceptance (parameter)

Difference of voltage angles of buses *n* and *m* (variable)

Market clearing considering network

$$\underset{p^{\mathrm{G1}},p^{\mathrm{G2}},p^{\mathrm{D1}},p^{\mathrm{D2}},\theta^{\mathrm{N1}},\theta^{\mathrm{N2}},\theta^{\mathrm{N3}}}{\text{Maximize}} \quad SW = [40p^{\mathrm{D1}} + 35p^{\mathrm{D1}}] - [12p^{\mathrm{G1}} + 20p^{\mathrm{G2}}]$$

subject to:

$$0 \le p^{\mathrm{D1}} \le 100$$

$$0 \le p^{\mathrm{D2}} \le 50$$

$$0 \le p^{\mathrm{G1}} \le 100$$

$$0 \le p^{\mathrm{G2}} \le 80$$

$$p^{\mathrm{G1}} - 500(\theta^{\mathrm{N1}} - \theta^{\mathrm{N2}}) - 500(\theta^{\mathrm{N1}} - \theta^{\mathrm{N3}}) = 0 \quad : \lambda^{\mathrm{N1}}$$

$$p^{\mathrm{G2}} - p^{\mathrm{D1}} - 500(\theta^{\mathrm{N2}} - \theta^{\mathrm{N1}}) - 500(\theta^{\mathrm{N2}} - \theta^{\mathrm{N3}}) = 0 \quad : \lambda^{\mathrm{N2}}$$

$$-p^{\mathrm{D2}} - 500(\theta^{\mathrm{N3}} - \theta^{\mathrm{N1}}) - 500(\theta^{\mathrm{N3}} - \theta^{\mathrm{N2}}) = 0 \quad : \lambda^{\mathrm{N3}}$$

$$-100 \le 500(\theta^{\mathrm{N1}} - \theta^{\mathrm{N2}}) \le 100$$

$$-100 \le 500(\theta^{\mathrm{N1}} - \theta^{\mathrm{N3}}) \le 100$$

$$-100 \le 500(\theta^{\mathrm{N2}} - \theta^{\mathrm{N3}}) \le 100$$

$$\theta^{\mathrm{N1}} = 0$$

Market clearing considering network

$$\underset{p^{G1},p^{G2},p^{D1},p^{D2},\theta^{N1},\theta^{N2},\theta^{N3}}{\text{Maximize}} \quad SW = [40p^{D1} + 35p^{D1}] - [12p^{G1} + 20p^{G2}]$$

subject to:

$$0 \leq p^{D1} \leq 100$$

$$0 \leq p^{D2} \leq 50$$

$$0 \leq p^{G1} \leq 100$$

$$0 \leq p^{G2} \leq 80$$

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$$p^{G2} - p^{D1} - 500(\theta^{N2} - \theta^{N1}) - 500(\theta^{N2} - \theta^{N3}) = 0 \quad : \lambda^{N2}$$

$$-p^{D2} - 500(\theta^{N3} - \theta^{N1}) - 500(\theta^{N3} - \theta^{N2}) = 0 \quad : \lambda^{N3}$$

Power balance at each bus

$$-100 \leq 500(\theta^{N1} - \theta^{N2}) \leq 100$$

$$-100 \leq 500(\theta^{N1} - \theta^{N3}) \leq 100$$

$$-100 \leq 500(\theta^{N2} - \theta^{N3}) \leq 100$$

Capacity of each transmission line

$$\theta^{N1} = 0$$ → Reference bus

Market clearing considering network

Compact form:

Market clearing considering network

Compact form:

$$\underset{p_g^{\mathrm{G}}, p_d^{\mathrm{D}}, \theta_n}{\text{Maximize}} \quad SW = \sum_d U_d \, p_d^{\mathrm{D}} - \sum_g C_g \, p_g^{\mathrm{G}}$$

subject to:

$$0 \leq p_d^{\mathrm{D}} \leq \overline{P}_d^{\mathrm{D}} \quad \forall d$$

$$0 \leq p_g^{\mathrm{G}} \leq \overline{P}_g^{\mathrm{G}} \quad \forall g$$

$$\sum_{d \in \Psi_n} p_d^{\mathrm{D}} + \sum_{m \in \Omega_n} B_{n,m}(\theta_n - \theta_m) - \sum_{g \in \Psi_n} p_g^{\mathrm{G}} = 0 \qquad : \lambda_n \quad \forall n$$

$$-F_{n,m} \leq B_{n,m}(\theta_n - \theta_m) \leq F_{n,m} \quad \forall n, \forall m$$

$$\theta_{ref} = 0$$

Market clearing considering network

Compact form:

$$\underset{p_g^{\mathrm{G}},p_d^{\mathrm{D}},\theta_n}{\text{Maximize}} \quad SW = \sum_d U_d \, p_d^{\mathrm{D}} - \sum_g C_g \, p_g^{\mathrm{G}}$$

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All demands located at bus *n*

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$$-F_{n,m} \leq B_{n,m}(\theta_n - \theta_m) \leq F_{n,m} \quad \forall n, \forall m$$

$$\theta_{ref} = 0$$

All buses *m* connected to bus *n* through transmission lines

Market clearing considering network

Compact form:

$$\underset{p_g^{\mathrm{G}}, p_d^{\mathrm{D}}, \theta_n}{\text{Maximize}} \quad SW = \sum_d U_d \, p_d^{\mathrm{D}} - \sum_g C_g \, p_g^{\mathrm{G}}$$

subject to:

$$0 \leq p_d^{\mathrm{D}} \leq \overline{P}_d^{\mathrm{D}} \quad \forall d$$

$$0 \leq p_g^{\mathrm{G}} \leq \overline{P}_g^{\mathrm{G}} \quad \forall g$$

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$$-F_{n,m} \leq B_{n,m}(\theta_n - \theta_m) \leq F_{n,m} \quad \forall n, \forall m$$

$$\theta_{ref} = 0$$

All generators located at bus *n*

Market clearing considering network

Compact form:

$$\underset{p_g^{\mathrm{G}}, p_d^{\mathrm{D}}, \theta_n}{\text{Maximize}} \quad SW = \sum_d U_d \, p_d^{\mathrm{D}} - \sum_g C_g \, p_g^{\mathrm{G}}$$

subject to:

$$0 \leq p_d^{\mathrm{D}} \leq \overline{P}_d^{\mathrm{D}} \quad \forall d$$

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$$\sum_{d \in \Psi_n} p_d^{\mathrm{D}} + \sum_{m \in \Omega_n} B_{n,m}(\theta_n - \theta_m) - \sum_{g \in \Psi_n} p_g^{\mathrm{G}} = 0 \quad : \lambda_n \quad \forall n$$

$$- F_{n,m} \leq B_{n,m}(\theta_n - \theta_m) \leq F_{n,m} \quad \forall n, \forall m$$

$$\theta_{ref} = 0$$

Nodal price at bus *n*
(locational marginal price, LMP)

Market clearing considering network

Compact form:

$$\underset{p_g^{\mathrm{G}}, p_d^{\mathrm{D}}, \theta_n}{\text{Maximize}} \quad SW = \sum_d U_d \, p_d^{\mathrm{D}} - \sum_g C_g \, p_g^{\mathrm{G}}$$

subject to:

$$0 \le p_d^{\mathrm{D}} \le \overline{P}_d^{\mathrm{D}} \quad \forall d$$

$$0 \le p_g^{\mathrm{G}} \le \overline{P}_g^{\mathrm{G}} \quad \forall g$$

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$$-F_{n,m} \le B_{n,m}(\theta_n - \theta_m) \le F_{n,m} \quad \forall n, \forall m$$

$$\theta_{ref} = 0$$

Capacity of line connecting bus *n* to bus *m*

Market clearing considering network

Compact form:

$$\underset{p_g^{\mathrm{G}}, p_d^{\mathrm{D}}, \theta_n}{\text{Maximize}} \quad SW = \sum_d U_d \, p_d^{\mathrm{D}} - \sum_g C_g \, p_g^{\mathrm{G}}$$

subject to:

$$0 \leq p_d^{\mathrm{D}} \leq \overline{P}_d^{\mathrm{D}} \quad \forall d$$

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$$\sum_{d \in \Psi_n} p_d^{\mathrm{D}} + \sum_{m \in \Omega_n} B_{n,m}(\theta_n - \theta_m) - \sum_{g \in \Psi_n} p_g^{\mathrm{G}} = 0 \qquad : \lambda_n \quad \forall n$$

$$-F_{n,m} \leq B_{n,m}(\theta_n - \theta_m) \leq F_{n,m} \quad \forall n, \forall m$$

$$\theta_{ref} = 0$$

Voltage angle at the reference bus

Market-clearing problem: primal optimization

$$\underset{p_g^{\mathrm{G}} \geq 0,\; p_d^{\mathrm{D}} \geq 0,\; \theta_n}{\text{Maximize}} \quad \sum_d U_d \, p_d^{\mathrm{D}} - \sum_g C_g \, p_g^{\mathrm{G}}$$

subject to:

$$p_d^{\mathrm{D}} \leq \overline{P}_d^{\mathrm{D}} \quad : \mu_d^{\mathrm{D}} \quad \forall d$$

$$p_g^{\mathrm{G}} \leq \overline{P}_g^{\mathrm{G}} \quad : \mu_g^{\mathrm{G}} \quad \forall g$$

$$\sum_{d \in \Psi_n} p_d^{\mathrm{D}} + \sum_{m \in \Omega_n} B_{n,m}(\theta_n - \theta_m) - \sum_{g \in \Psi_n} p_g^{\mathrm{G}} = 0 \quad : \lambda_n \quad \forall n$$

$$-F_{n,m} \leq B_{n,m}(\theta_n - \theta_m) \leq F_{n,m} \quad : \underline{\eta}_{n,m}, \overline{\eta}_{n,m} \quad \forall n, \forall m \in \Omega_n$$

$$\theta_{(n=ref)} = 0 \quad : \gamma$$

Market-clearing problem: dual optimization

$$\underset{\mu_d^{\mathrm{D}}\geq 0,\ \mu_g^{\mathrm{G}}\geq 0,\ \underline{\eta}_{n,m}\geq 0,\ \overline{\eta}_{n,m}\geq 0,\ \lambda_n,\ \gamma}{\text{Minimize}} \quad \sum_d \mu_d^{\mathrm{D}}\ \overline{P}_d^{\mathrm{D}} + \sum_g \mu_g^{\mathrm{G}}\ \overline{P}_g^{\mathrm{G}} + \sum_{n,(m\in\Omega_n)} F_{n,m}(\underline{\eta}_{n,m} + \overline{\eta}_{n,m})$$

subject to:

$$-U_d + \mu_d^{\mathrm{D}} + \lambda_{n\in\Psi_d} \geq 0 \quad : p_d^{\mathrm{D}} \quad \forall d$$

$$C_g + \mu_g^{\mathrm{G}} - \lambda_{n\in\Psi_g} \geq 0 \quad : p_g^{\mathrm{G}} \quad \forall g$$

$$\sum_{m\in\Omega_n} B_{n,m}(\lambda_n - \lambda_m + \overline{\eta}_{n,m} - \overline{\eta}_{m,n} - \underline{\eta}_{n,m} + \underline{\eta}_{m,n}) + \gamma = 0 \quad : \theta_n \quad n = ref$$

$$\sum_{m\in\Omega_n} B_{n,m}(\lambda_n - \lambda_m + \overline{\eta}_{n,m} - \overline{\eta}_{m,n} - \underline{\eta}_{n,m} + \underline{\eta}_{m,n}) = 0 \quad : \theta_n \quad \forall n\ /\ ref$$

Market-clearing problem: dual optimization

$$\underset{\mu_d^{\mathrm{D}}\geq 0,\ \mu_g^{\mathrm{G}}\geq 0,\ \underline{\eta}_{n,m}\geq 0,\ \overline{\eta}_{n,m}\geq 0,\ \lambda_n,\ \gamma}{\text{Minimize}} \quad \sum_d \mu_d^{\mathrm{D}}\ \overline{P}_d^{\mathrm{D}} + \sum_g \mu_g^{\mathrm{G}}\ \overline{P}_g^{\mathrm{G}} + \sum_{n,(m\in\Omega_n)} F_{n,m}(\underline{\eta}_{n,m} + \overline{\eta}_{n,m})$$

subject to:

$$-U_d + \mu_d^{\mathrm{D}} + \lambda_{n\in\Psi_d} \geq 0 \quad : p_d^{\mathrm{D}} \quad \forall d$$

$$C_g + \mu_g^{\mathrm{G}} - \lambda_{n\in\Psi_g} \geq 0 \quad : p_g^{\mathrm{G}} \quad \forall g$$

$$\sum_{m\in\Omega_n} B_{n,m}(\lambda_n - \lambda_m + \overline{\eta}_{n,m} - \overline{\eta}_{m,n} - \underline{\eta}_{n,m} + \underline{\eta}_{m,n}) + \gamma = 0 \quad : \theta_n \quad n = ref$$

$$\sum_{m\in\Omega_n} B_{n,m}(\lambda_n - \lambda_m + \overline{\eta}_{n,m} - \overline{\eta}_{m,n} - \underline{\eta}_{n,m} + \underline{\eta}_{m,n}) = 0 \quad : \theta_n \quad \forall n \ / \ ref$$

Exercise 1: Derive this formulation yourself!

Market-clearing problem: dual optimization

$$\underset{\mu_d^{\mathrm{D}}\geq 0,\ \mu_g^{\mathrm{G}}\geq 0,\ \underline{\eta}_{n,m}\geq 0,\ \overline{\eta}_{n,m}\geq 0,\ \lambda_n,\ \gamma}{\text{Minimize}} \quad \sum_d \mu_d^{\mathrm{D}}\ \overline{P}_d^{\mathrm{D}} + \sum_g \mu_g^{\mathrm{G}}\ \overline{P}_g^{\mathrm{G}} + \sum_{n,(m\in\Omega_n)} F_{n,m}(\underline{\eta}_{n,m} + \overline{\eta}_{n,m})$$

subject to:

$$-U_d + \mu_d^{\mathrm{D}} + \lambda_{n\in\Psi_d} \geq 0 \quad : p_d^{\mathrm{D}} \quad \forall d$$

$$C_g + \mu_g^{\mathrm{G}} - \lambda_{n\in\Psi_g} \geq 0 \quad : p_g^{\mathrm{G}} \quad \forall g$$

$$\sum_{m\in\Omega_n} B_{n,m}(\lambda_n - \lambda_m + \overline{\eta}_{n,m} - \overline{\eta}_{m,n} - \underline{\eta}_{n,m} + \underline{\eta}_{m,n}) + \gamma = 0 \quad : \theta_n \quad n = ref$$

$$\sum_{m\in\Omega_n} B_{n,m}(\lambda_n - \lambda_m + \overline{\eta}_{n,m} - \overline{\eta}_{m,n} - \underline{\eta}_{n,m} + \underline{\eta}_{m,n}) = 0 \quad : \theta_n \quad \forall n\ /\ ref$$

Exercise 1: Derive this formulation yourself!

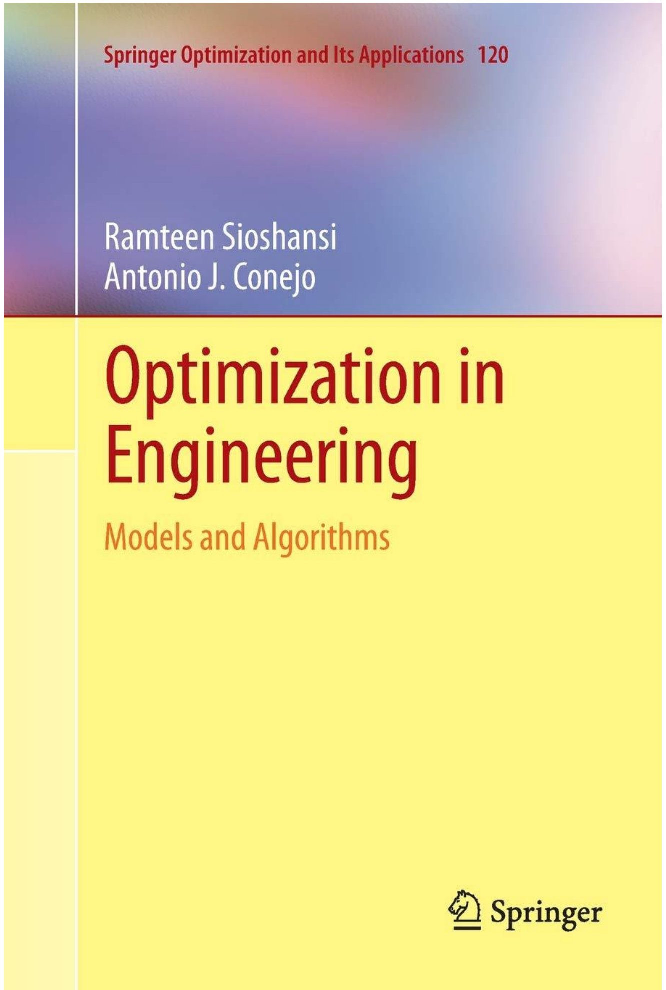
How to derive a dual optimization? Next session!

Thanks for your attention!

Email: jalal@dtu.dk

How to derive optimality conditions and dual problem of a linear optimization problem?

References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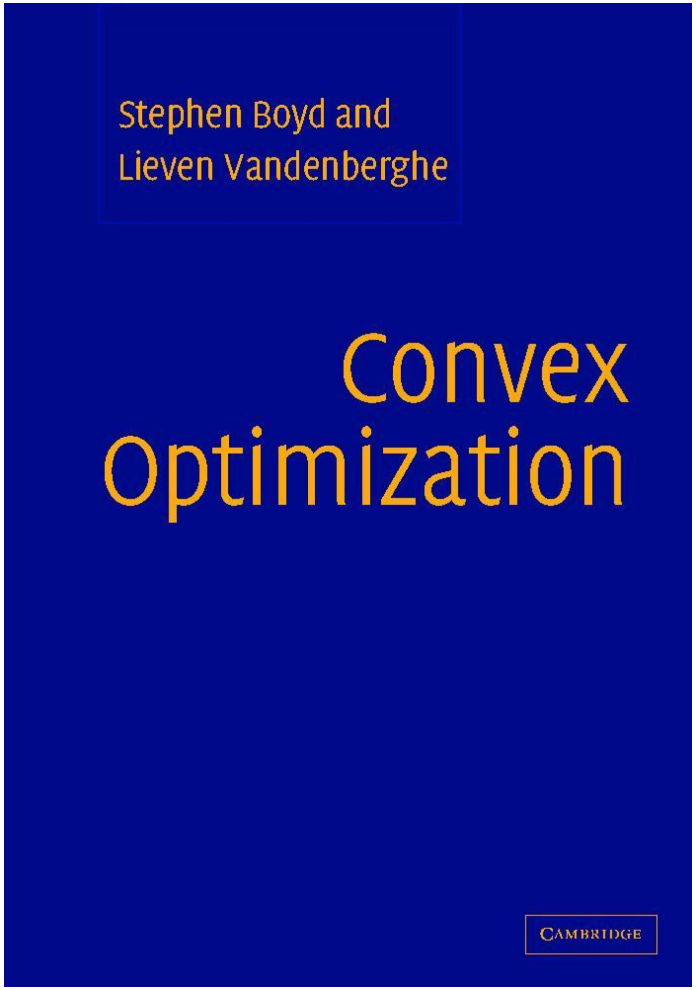
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A mathematician on a mission

Mathematical analysis | Operations analysis | Mathematics

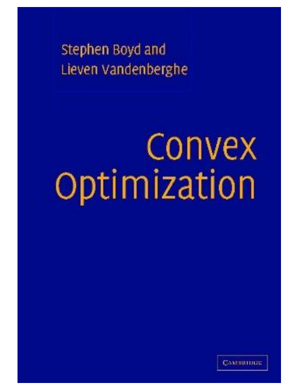
TUESDAY 27 NOV 18 | By Morten Andersen

Stanford University Professor Stephen Boyd applies convex optimization to a wide range of engineering problems. With astounding results.

"DTU should teach a course on convex optimization. And all students should be obliged to take it!"

HC Ørsted lectures

Twice a year, DTU invites prominent foreign researchers to lecture on their work, research findings, and the prospects within their field of research at the so-called Ørsted Lectures. The lectures are open to all.



How to derive Lagrangian function?

$$\underset{x}{\text{Minimize}} \quad f(x)$$

$$\text{subject to:}$$

$$h(x) = 0 \quad : \quad \lambda$$

$$g(x) \leq 0 \quad : \quad \mu$$

This is a standard form of an optimization problem!

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$$\mathcal{L}(x, \lambda, \mu) = f(x) + \lambda^\top h(x) + \mu^\top g(x)$$

How to derive optimality conditions?

Original (primal) problem

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Lagrangian function

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \lambda^\top h(x) + \mu^\top g(x)$$

$$\frac{\partial \mathcal{L}(x, \lambda, \mu)}{\partial x} = 0$$

$$h(x) = 0$$

$$0 \leq -g(x) \perp \mu \geq 0$$

$$\lambda \in \text{free}$$

Optimality
Karush–Kuhn–Tucker (KKT)
conditions

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Optimality
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Complementarity condition

Example

Let us consider the following linear optimization problem:

$$\underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad 18x_1 + 8x_2 + 12x_3 + 16x_4$$

subject to:

$$\frac{2}{3}x_1 + 2x_2 + x_3 \geq 1 \quad : \quad \mu_1$$

$$x_1 + x_2 + x_4 \geq 1 \quad : \quad \mu_2$$

$$x_1 \geq 0 \quad : \quad \mu_3$$

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Example

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Four primal variables

subject to:

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Six dual variables, one per constraint

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Six dual variables, one per constraint

Recall:
When we derive Lagrangian function, the inequality constraints should be in form of

$$g(x) \leq 0$$

Example

Original (primal) problem

$$\underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad 18x_1 + 8x_2 + 12x_3 + 16x_4$$

subject to:

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Lagrangian function

$$\begin{aligned}\mathcal{L}(\mathbf{x}, \mu) = {} & 18x_1 + 8x_2 + 12x_3 + 16x_4 \\ & - \mu_1\left(\frac{2}{3}x_1 + 2x_2 + x_3 - 1\right) \\ & - \mu_2(x_1 + x_2 + x_4 - 1) \\ & - \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4\end{aligned}$$

Example

Lagrangian function

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Optimality KKT conditions

$$\frac{\partial \mathcal{L}(\mathbf{x}, \mu)}{\partial x_1} = 18 - \frac{2}{3}\mu_1 - \mu_2 - \mu_3 = 0$$

$$\frac{\partial \mathcal{L}(\mathbf{x}, \mu)}{\partial x_2} = 8 - 2\mu_1 - \mu_2 + \mu_4 = 0$$

$$\frac{\partial \mathcal{L}(\mathbf{x}, \mu)}{\partial x_3} = 12 - \mu_1 - \mu_5 = 0$$

$$\frac{\partial \mathcal{L}(\mathbf{x}, \mu)}{\partial x_4} = 16 - \mu_2 - \mu_6 = 0$$

$$0 \leq (\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \perp \mu_1 \geq 0$$

$$0 \leq (x_1 + x_2 + x_4 - 1) \perp \mu_2 \geq 0$$

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Example

Lagrangian function

$$\mathcal{L}(\mathbf{x},\mu) = 18x_1 + 8x_2 + 12x_3 + 16x_4 - \mu_1(\frac{2}{3}x_1 + 2x_2 + x_3 - 1) - \mu_2(x_1 + x_2 + x_4 - 1) - \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4$$

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Can we write KKT conditions in a more compact way?

Example

Lagrangian function

$$\begin{aligned}\mathcal{L}(\mathbf{x},\mu) = {} & 18x_1 + 8x_2 + 12x_3 + 16x_4 \\ & - \mu_1(\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \\ & - \mu_2(x_1 + x_2 + x_4 - 1) \\ & - \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4\end{aligned}$$

Optimality KKT conditions

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$$0 \leq (\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \perp \mu_1 \geq 0$$

$$0 \leq (x_1 + x_2 + x_4 - 1) \perp \mu_2 \geq 0$$

$$0 \leq x_1 \perp \mu_3 \geq 0$$

$$0 \leq -x_2 \perp \mu_4 \geq 0$$

$$0 \leq x_3 \perp \mu_5 \geq 0$$

$$0 \leq x_4 \perp \mu_6 \geq 0$$

Can we write KKT conditions in a more compact way? Yes!

Example

Lagrangian function

$$\mathcal{L}(\mathbf{x},\mu) = 18x_1 + 8x_2 + 12x_3 + 16x_4$$
$$- \mu_1(\frac{2}{3}x_1 + 2x_2 + x_3 - 1)$$
$$- \mu_2(x_1 + x_2 + x_4 - 1)$$
$$- \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4$$

Optimality KKT conditions

$$\frac{\partial \mathcal{L}(\mathbf{x},\mu)}{\partial x_1} = 18 - \frac{2}{3}\mu_1 - \mu_2 - \mu_3 = 0$$
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$$0 \leq x_1 \perp \mu_3 \geq 0$$
$$0 \leq -x_2 \perp \mu_4 \geq 0$$
$$0 \leq x_3 \perp \mu_5 \geq 0$$
$$0 \leq x_4 \perp \mu_6 \geq 0$$

For example, we can merge these two conditions to get rid of dual variable \mu_{3} corresponding to the non-negativity condition of x_{1}, i.e.,

$$0 \leq x_1 \perp (18 - \frac{2}{3}\mu_1 - \mu_2) \geq 0$$

Example

Eventually, the optimality KKT conditions are

Original (primal) problem

$$\underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad 18x_1 + 8x_2 + 12x_3 + 16x_4$$

subject to:

$$\frac{2}{3}x_1 + 2x_2 + x_3 \geq 1 \quad : \quad \mu_1$$

$$x_1 + x_2 + x_4 \geq 1 \quad : \quad \mu_2$$

$$x_1 \geq 0$$

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⇨

Optimality KKT conditions

$$0 \leq (\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \perp \mu_1 \geq 0$$

$$0 \leq (x_1 + x_2 + x_4 - 1) \perp \mu_2 \geq 0$$

$$0 \leq x_1 \perp (18 - \frac{2}{3}\mu_1 - \mu_2) \geq 0$$

$$0 \leq -x_2 \perp (-8 + 2\mu_1 + \mu_2) \geq 0$$

$$0 \leq x_3 \perp (12 - \mu_1) \geq 0$$

$$0 \leq x_4 \perp (16 - \mu_2) \geq 0$$

Example

How to write a code to directly solve KKT conditions (as a system of equations)?

Option 1: Solve using PATH solver (http://pages.cs.wisc.edu/~ferris/path.html)

Option 2: Define an auxiliary objective function (e.g., minimize 1), consider KKT conditions as the constraints, and then solve the resulting optimization problem using a non-linear solver (nonlinearity comes from complementarity conditions) ---- we will discuss later in this course how to linearize the complementarity conditions using auxiliary binary (0/1) variables!

How to derive dual problem?

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Discussion:

Why is it appealing to derive dual problem?

How to derive dual problem?

Recall that

Original (primal) problem

$$\begin{aligned} &\underset{x}{\text{Minimize}} \quad f(x) \\ &\text{subject to:} \\ &h(x) = 0 \quad : \quad \lambda \\ &g(x) \leq 0 \quad : \quad \mu \end{aligned}$$

Lagrangian function

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \lambda^\top h(x) + \mu^\top g(x)$$

Step 1: derive "dual function" as $\underset{x}{\text{Minimize}} \quad \mathcal{L}(x, \lambda, \mu)$

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Lagrangian function

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \lambda^\top h(x) + \mu^\top g(x)$$

Step 1: derive "dual function" as $\underset{x}{\text{Minimize}} \quad \mathcal{L}(x, \lambda, \mu)$

- Dual function is an unconstrained optimization problem. For arbitrarily given values of dual variables (\mu should be non-negative), the dual function minimizes the (relaxed) Lagrangian function. Primal variables are the only variables to be optimized.
- Why "relaxed"? Because constraints in the original primal problem are relaxed, and the fixed dual variables in the dual function "penalize" the violation of relaxed constraints.
- The optimal value of the dual function provides a "**lower bound**" for the optimal value of objective function of the original primal problem.
- More info? Watch this short video: https://www.youtube.com/watch?v=4OifjG2kIJQ

How to derive dual problem?

Recall that

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Lagrangian function

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \lambda^\top h(x) + \mu^\top g(x)$$

Step 2: derive "dual problem" which provides the best possible lower bound, i.e.,

$$\underset{\lambda \in \text{free} \ , \ \mu \geq 0}{\text{Maximize}} \quad \underbrace{\underset{x}{\text{Minimize}} \quad \mathcal{L}(x, \lambda, \mu)}_{\text{dual function (i.e., lower bound)}}$$

dual function
(i.e., lower bound)

Example

Recall our previous example

Original (primal) problem

$$\underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad 18x_1 + 8x_2 + 12x_3 + 16x_4$$

subject to:

$$\frac{2}{3}x_1 + 2x_2 + x_3 \geq 1 \quad : \quad \mu_1$$

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Lagrangian function

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Example

Recall our previous example

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Dual problem

$$\underset{\mu_1,\ldots,\mu_6 \geq 0}{\text{Maximize}} \left\{ \begin{aligned} \underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad & 18x_1 + 8x_2 + 12x_3 + 16x_4 \\ & - \mu_1(\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \\ & - \mu_2(x_1 + x_2 + x_4 - 1) \\ & - \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4 \end{aligned} \right\}$$

Example

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Lagrangian function

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These two terms (-1 times -\mu) are constants in the inner (i.e., minimization) problem, but variables in the outer (i.e., maximization) problem!

Dual problem

$$\underset{\mu_1,\ldots,\mu_6 \geq 0}{\text{Maximize}} \left\{ \begin{aligned} \underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad & 18x_1 + 8x_2 + 12x_3 + 16x_4 \\ & - \mu_1(\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \\ & - \mu_2(x_1 + x_2 + x_4 - 1) \\ & - \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4 \end{aligned} \right\}$$

Example

Dual problem:

$$\underset{\mu_1,\ldots,\mu_6\geq 0}{\text{Maximize}} \left\{ \begin{aligned} \underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad & 18x_1 + 8x_2 + 12x_3 + 16x_4 \\ & - \mu_1(\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \\ & - \mu_2(x_1 + x_2 + x_4 - 1) \\ & - \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4 \end{aligned} \right\}$$

Example

Dual problem:

$$\underset{\mu_1,\ldots,\mu_6 \geq 0}{\text{Maximize}} \left\{ \begin{array}{l} \underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad 18x_1 + 8x_2 + 12x_3 + 16x_4 \\ \qquad - \mu_1(\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \\ \qquad - \mu_2(x_1 + x_2 + x_4 - 1) \\ \qquad - \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4 \end{array} \right\}$$

$$\Downarrow$$

$$\underset{\mu_1,\ldots,\mu_6 \geq 0}{\text{Maximize}} \quad \mu_1 + \mu_2$$

subject to:

$$18 - \frac{2}{3}\mu_1 - \mu_2 - \mu_3 = 0$$

$$8 - 2\mu_1 - \mu_2 + \mu_4 = 0$$

$$12 - \mu_1 - \mu_5 = 0$$

$$16 - \mu_2 - \mu_6 = 0$$

Example

Dual problem:

$$\underset{\mu_1,\ldots,\mu_6\geq 0}{\text{Maximize}} \left\{ \begin{aligned} \underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad & 18x_1 + 8x_2 + 12x_3 + 16x_4 \\ & - \mu_1(\tfrac{2}{3}x_1 + 2x_2 + x_3 - 1) \\ & - \mu_2(x_1 + x_2 + x_4 - 1) \\ & - \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4 \end{aligned} \right\}$$

⇓

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Can we write the dual problem in a more compact way?

Example

Dual problem:

$$\underset{\mu_1,\ldots,\mu_6\geq 0}{\text{Maximize}} \left\{ \begin{aligned} \underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad & 18x_1 + 8x_2 + 12x_3 + 16x_4 \\ & - \mu_1(\frac{2}{3}x_1 + 2x_2 + x_3 - 1) \\ & - \mu_2(x_1 + x_2 + x_4 - 1) \\ & - \mu_3 x_1 + \mu_4 x_2 - \mu_5 x_3 - \mu_6 x_4 \end{aligned} \right\}$$

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Can we write the dual problem in a more compact way? Yes!

Note:

- Dual variables \mu_{3} to \mu_{6} are isolated, since they do not appear in the objective function, and do not link constraints!
- We also know that they are non-negative.
- So, we can get rid of them by converting equalities to inequalities.

Example

Dual problem:

$$\underset{\mu_1,\ldots,\mu_6 \geq 0}{\text{Maximize}} \quad \mu_1 + \mu_2$$

subject to:

$$18 - \frac{2}{3}\mu_1 - \mu_2 - \mu_3 = 0$$

$$8 - 2\mu_1 - \mu_2 + \mu_4 = 0$$

$$12 - \mu_1 - \mu_5 = 0$$

$$16 - \mu_2 - \mu_6 = 0$$

$\Rightarrow$

$$\underset{\mu_1,\mu_2 \geq 0}{\text{Maximize}} \quad \mu_1 + \mu_2$$

subject to:

$$18 - \frac{2}{3}\mu_1 - \mu_2 \geq 0$$

$$8 - 2\mu_1 - \mu_2 \leq 0$$

$$12 - \mu_1 \geq 0$$

$$16 - \mu_2 \geq 0$$

Example

Dual problem:

$$\underset{\mu_1,\ldots,\mu_6 \geq 0}{\text{Maximize}} \quad \mu_1 + \mu_2$$

subject to:

$$18 - \frac{2}{3}\mu_1 - \mu_2 - \mu_3 = 0$$

$$8 - 2\mu_1 - \mu_2 + \mu_4 = 0$$

$$12 - \mu_1 - \mu_5 = 0$$

$$16 - \mu_2 - \mu_6 = 0$$

$\Rightarrow$

$$\underset{\mu_1,\mu_2 \geq 0}{\text{Maximize}} \quad \mu_1 + \mu_2$$

subject to:

$$18 - \frac{2}{3}\mu_1 - \mu_2 \geq 0$$

$$8 - 2\mu_1 - \mu_2 \leq 0$$

$$12 - \mu_1 \geq 0$$

$$16 - \mu_2 \geq 0$$

Example

Primal problem

Two options, both are equivalent:

Option 1

$$\underset{x_1,x_2,x_3,x_4}{\text{Minimize}} \quad 18x_1 + 8x_2 + 12x_3 + 16x_4$$

subject to:

$$\frac{2}{3}x_1 + 2x_2 + x_3 \geq 1 \quad : \quad \mu_1$$

$$x_1 + x_2 + x_4 \geq 1 \quad : \quad \mu_2$$

$$x_1 \geq 0 \quad : \quad \mu_3$$

$$-x_2 \geq 0 \quad : \quad \mu_4$$

$$x_3 \geq 0 \quad : \quad \mu_5$$

$$x_4 \geq 0 \quad : \quad \mu_6$$

Dual problem

$$\underset{\mu_1,\ldots,\mu_6 \geq 0}{\text{Maximize}} \quad \mu_1 + \mu_2$$

subject to:

$$18 - \frac{2}{3}\mu_1 - \mu_2 - \mu_3 = 0$$

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Example

Primal problem

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Option 1

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$$x_1 \geq 0 \quad : \quad \mu_3$$

$$-x_2 \geq 0 \quad : \quad \mu_4$$

$$x_3 \geq 0 \quad : \quad \mu_5$$

$$x_4 \geq 0 \quad : \quad \mu_6$$

Dual problem

$$\underset{\mu_1,\dots,\mu_6 \geq 0}{\text{Maximize}} \quad \mu_1 + \mu_2$$

subject to:

$$18 - \frac{2}{3}\mu_1 - \mu_2 - \mu_3 = 0$$

$$8 - 2\mu_1 - \mu_2 + \mu_4 = 0$$

$$12 - \mu_1 - \mu_5 = 0$$

$$16 - \mu_2 - \mu_6 = 0$$

Option 2 (preferred, due to less number of variables/constraints)

$$\underset{x_1,x_3,x_4 \geq 0,\ x_2 \leq 0}{\text{Minimize}} \quad 18x_1 + 8x_2 + 12x_3 + 16x_4$$

subject to:

$$\frac{2}{3}x_1 + 2x_2 + x_3 \geq 1 \quad : \quad \mu_1$$

$$x_1 + x_2 + x_4 \geq 1 \quad : \quad \mu_2$$

$$\underset{\mu_1,\mu_2 \geq 0}{\text{Maximize}} \quad \mu_1 + \mu_2$$

subject to:

$$18 - \frac{2}{3}\mu_1 - \mu_2 \geq 0$$

$$8 - 2\mu_1 - \mu_2 \leq 0$$

$$12 - \mu_1 \geq 0$$

$$16 - \mu_2 \geq 0$$

Important points

- ✓ Number of variables in the primal problem = Number of constraints in the dual problem

- ✓ Number of constraints in the primal problem = Number of variables in the dual problem

- ✓ Dual problem of the dual problem is the primal problem!

- ✓ Dual variables of the dual problem are the primal variables!

Important points

✓ **Weak duality theorem**:

The value of objective function of the dual problem at any point of its feasible region is *lower than or equal to* that of the primal problem at any point of its feasible region.

In our example:

$$18x_1 + 8x_2 + 12x_3 + 16x_4 \geq \mu_1 + \mu_2$$

Important points

✓ **Weak duality theorem**:

The value of objective function of the dual problem at any point of its feasible region is *lower than or equal to* that of the primal problem at any point of its feasible region.

In our example:

$$\underbrace{18x_1 + 8x_2 + 12x_3 + 16x_4}_{\text{The value of the objective function of the primal problem}} \geq \underbrace{\mu_1 + \mu_2}_{\text{The value of the objective function of the dual problem}}$$

Important points

✓ **Weak duality theorem**:

The value of objective function of the dual problem at any point of its feasible region is *lower than or equal to* that of the primal problem at any point of its feasible region.

In our example:

$$\underbrace{18x_1 + 8x_2 + 12x_3 + 16x_4}_{\text{The value of the objective function of the primal problem}} \geq \underbrace{\mu_1 + \mu_2}_{\text{The value of the objective function of the dual problem}}$$

Important points

✓ **Strong duality theorem**:

In the optimal point, if Slater's condition holds, the value of objective function of the dual problem is *equal to* that of the primal problem.

In our example [note that superscript * denotes the optimal value]:

$$18x_1^* + 8x_2^* + 12x_3^* + 16x_4^* = \mu_1^* + \mu_2^*$$

The value of the objective function of the primal problem (left-hand side)

The value of the objective function of the dual problem (right-hand side)

Important points

✓ **Strong duality theorem**:

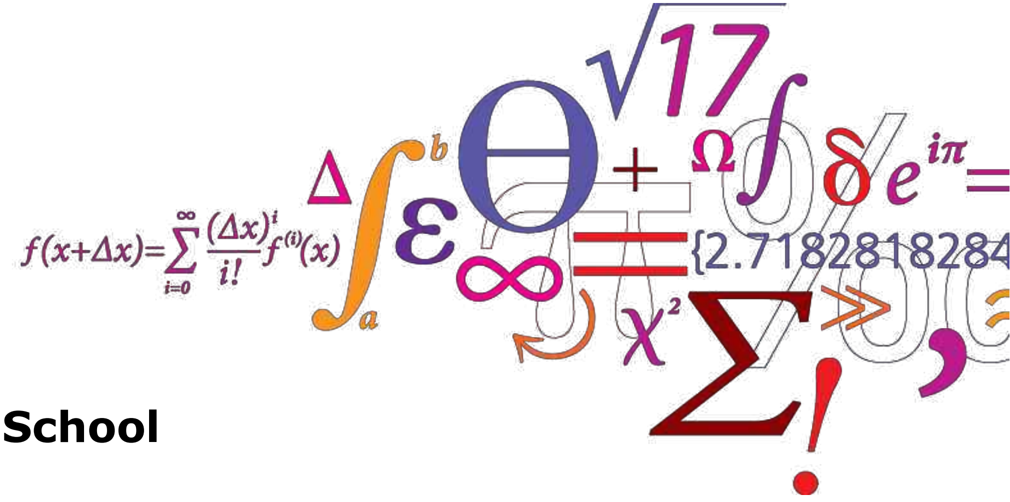
In the optimal point, if Slater's condition holds, the value of objective function of the dual problem is ***equal to*** that of the primal problem.

In our example [note that superscript * denotes the optimal value]:

$$18x_1^* + 8x_2^* + 12x_3^* + 16x_4^* = \mu_1^* + \mu_2^*$$

The value of the objective function of the primal problem | **The value of the objective function of the dual problem**

Lecture 2: Market clearing as an equilibrium problem



Jalal Kazempour

The 6th KIOS Graduate Training School

September 2, 2024

Recap

Market clearing: a simple example

An extended example: two generators (G1 and G2) and two elastic demands (D1 and D2)

Capacity: 100 MW
Offer price: $12/MWh

Capacity: 80 MW
Offer price: $20/MWh

Demand D1

Maximum load: 100 MW
Bid price: $40/MWh

Demand D2

Maximum load: 50 MW
Bid price: $35/MWh

Market outcomes:

- Productions of G1 and G2: **100 MW and 50 MW**
- Consumptions of D1 and D2: **100 MW and 50 MW**
- Market-clearing price: **[20-35] $/MWh**

Market clearing as an optimization problem

$$\underset{p^{G1},p^{G2},p^{D1},p^{D2}}{\text{Maximize}} \quad SW = [40p^{D1} + 35p^{D2}] - [12p^{G1} + 20p^{G2}] \tag{1a}$$

subject to:

$$0 \leq p^{D1} \leq 100 \tag{1b}$$

$$0 \leq p^{D2} \leq 50 \tag{1c}$$

$$0 \leq p^{G1} \leq 100 \tag{1d}$$

$$0 \leq p^{G2} \leq 80 \tag{1e}$$

$$p^{D1} + p^{D2} - p^{G1} - p^{G2} = 0 \qquad : \lambda \tag{1f}$$

Market clearing as an optimization problem

$$\underset{p^{G1},p^{G2},p^{D1},p^{D2}}{\text{Maximize}} \quad SW = \underbrace{[40p^{D1} + 35p^{D2}]}_{\text{Utility of demands}} - \underbrace{[12p^{G1} + 20p^{G2}]}_{\text{Cost of generators}} \tag{1a}$$

subject to:

Consumption limits:

$$0 \leq p^{D1} \leq 100 \tag{1b}$$

$$0 \leq p^{D2} \leq 50 \tag{1c}$$

Generation limits:

$$0 \leq p^{G1} \leq 100 \tag{1d}$$

$$0 \leq p^{G2} \leq 80 \tag{1e}$$

$$\underbrace{p^{D1} + p^{D2} - p^{G1} - p^{G2} = 0}_{\text{Power balance equality}} \quad : \lambda \tag{1f}$$

λ → Dual variable: market-clearing price

Discussion

Question:

How to make sure all market participants (i.e., G1, G2, D1 and D2) are satisfied with the market-clearing outcome, and would not deviate from it?

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Let's develop an optimization problem for each market participant!

Discussion

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Let's develop an optimization problem for each market participant!

Question:

What is the objective of each generator?

What is the objective of each elastic demand?

Discussion

Question:

How to make sure all market participants (i.e., G1, G2, D1 and D2) are satisfied with the market-clearing outcome, and would not deviate from it?

Let's develop an optimization problem for each market participant!

Question:

What is the objective of each generator? Profit maximization!

What is the objective of each elastic demand? Utility maximization!

Discussion

Question:

How to make sure all market participants (i.e., G1, G2, D1 and D2) are satisfied with the market-clearing outcome, and would not deviate from it?

Let's develop an optimization problem for each market participant!

Question:

What is the objective of each generator? Profit maximization!

What is the objective of each elastic demand? Utility maximization!

Question:

How to calculate a generator's profit?

How to calculate an elastic demand's utility?

Discussion

Question:

How to make sure all market participants (i.e., G1, G2, D1 and D2) are satisfied with the market-clearing outcome, and would not deviate from it?

Let's develop an optimization problem for each market participant!

Question:

What is the objective of each generator? Profit maximization!

What is the objective of each elastic demand? Utility maximization!

Question:

How to calculate a generator's profit? Production level x [market price – production cost]

How to calculate an elastic demand's utility? Consumption level x [bid price – market price]

Optimization problem for each market player

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$

subject to:

$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$

subject to:

$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

For generator G2:

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$

subject to:

$$0 \leq p^{G2} \leq 80 \quad : \underline{\mu}^{G2}, \overline{\mu}^{G2}$$

Optimization problem for each market player

For generator G1:

$$\underset{p^{\mathrm{G1}}}{\text{Maximize}} \quad p^{\mathrm{G1}}(\lambda - 12)$$

subject to:

$$0 \leq p^{\mathrm{G1}} \leq 100 \quad : \underline{\mu}^{\mathrm{G1}}, \overline{\mu}^{\mathrm{G1}}$$

For generator G2:

$$\underset{p^{\mathrm{G2}}}{\text{Maximize}} \quad p^{\mathrm{G2}}(\lambda - 20)$$

subject to:

$$0 \leq p^{\mathrm{G2}} \leq 80 \quad : \underline{\mu}^{\mathrm{G2}}, \overline{\mu}^{\mathrm{G2}}$$

For demand D1:

$$\underset{p^{\mathrm{D1}}}{\text{Maximize}} \quad p^{\mathrm{D1}}(40 - \lambda)$$

subject to:

$$0 \leq p^{\mathrm{D1}} \leq 100 \quad : \underline{\mu}^{\mathrm{D1}}, \overline{\mu}^{\mathrm{D1}}$$

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$

subject to:

$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

For generator G2:

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$

subject to:

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For demand D1:

$$\underset{p^{D1}}{\text{Maximize}} \quad p^{D1}(40 - \lambda)$$

subject to:

$$0 \leq p^{D1} \leq 100 \quad : \underline{\mu}^{D1}, \overline{\mu}^{D1}$$

For demand D2:

$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$

subject to:

$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$

subject to:

$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

For generator G2:

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$

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For demand D1:

$$\underset{p^{D1}}{\text{Maximize}} \quad p^{D1}(40 - \lambda)$$

subject to:

$$0 \leq p^{D1} \leq 100 \quad : \underline{\mu}^{D1}, \overline{\mu}^{D1}$$

For demand D2:

$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$

subject to:

$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

Market price λ is a given value (treated as a parameter) within each optimization problem!

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$
subject to:
$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

For generator G2:

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$
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For demand D1:

$$\underset{p^{D1}}{\text{Maximize}} \quad p^{D1}(40 - \lambda)$$
subject to:
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For demand D2:

$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$
subject to:
$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

Market price λ is a given value (treated as a parameter) within each optimization problem!

Question:
How do market players contribute to market price formation? Do we need an extra condition?

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$
subject to:
$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

For generator G2:

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$
subject to:
$$0 \leq p^{G2} \leq 80 \quad : \underline{\mu}^{G2}, \overline{\mu}^{G2}$$

For demand D1:

$$\underset{p^{D1}}{\text{Maximize}} \quad p^{D1}(40 - \lambda)$$
subject to:
$$0 \leq p^{D1} \leq 100 \quad : \underline{\mu}^{D1}, \overline{\mu}^{D1}$$

For demand D2:

$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$
subject to:
$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

Market price λ is a given value (treated as a parameter) within each optimization problem!

Question:
How do market players contribute to market price formation? Do we need an extra condition? Yes!

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$

subject to:

$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

For generator G2:

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$

subject to:

$$0 \leq p^{G2} \leq 80 \quad : \underline{\mu}^{G2}, \overline{\mu}^{G2}$$

For demand D1:

$$\underset{p^{D1}}{\text{Maximize}} \quad p^{D1}(40 - \lambda)$$

subject to:

$$0 \leq p^{D1} \leq 100 \quad : \underline{\mu}^{D1}, \overline{\mu}^{D1}$$

For demand D2:

$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$

subject to:

$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

Power balance equality:

$$p^{D1} + p^{D2} - p^{G1} - p^{G2} = 0 \quad : \lambda$$

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$

subject to:

$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

For generator G2:

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$

subject to:

$$0 \leq p^{G2} \leq 80 \quad : \underline{\mu}^{G2}, \overline{\mu}^{G2}$$

For demand D1:

$$\underset{p^{D1}}{\text{Maximize}} \quad p^{D1}(40 - \lambda)$$

subject to:

$$0 \leq p^{D1} \leq 100 \quad : \underline{\mu}^{D1}, \overline{\mu}^{D1}$$

For demand D2:

$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$

subject to:

$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

Power balance equality:

$$p^{D1} + p^{D2} - p^{G1} - p^{G2} = 0 \quad : \lambda$$

Question: From mathematical perspective, is the above power balance equality "equivalent" to the following "unconstrained" optimization problem? Why?

$$\underset{\lambda}{\text{Minimize}} \quad \lambda\left(p^{D1} + p^{D2} - p^{G1} - p^{G2}\right)$$

Optimization problem for each market player

For generator G1:

$$\underset{p^{\mathrm{G1}}}{\text{Maximize}} \quad p^{\mathrm{G1}}(\lambda - 12)$$

subject to:

$$0 \leq p^{\mathrm{G1}} \leq 100 \quad : \underline{\mu}^{\mathrm{G1}}, \overline{\mu}^{\mathrm{G1}}$$

For generator G2:

$$\underset{p^{\mathrm{G2}}}{\text{Maximize}} \quad p^{\mathrm{G2}}(\lambda - 20)$$

subject to:

$$0 \leq p^{\mathrm{G2}} \leq 80 \quad : \underline{\mu}^{\mathrm{G2}}, \overline{\mu}^{\mathrm{G2}}$$

For demand D1:

$$\underset{p^{\mathrm{D1}}}{\text{Maximize}} \quad p^{\mathrm{D1}}(40 - \lambda)$$

subject to:

$$0 \leq p^{\mathrm{D1}} \leq 100 \quad : \underline{\mu}^{\mathrm{D1}}, \overline{\mu}^{\mathrm{D1}}$$

For demand D2:

$$\underset{p^{\mathrm{D2}}}{\text{Maximize}} \quad p^{\mathrm{D2}}(35 - \lambda)$$

subject to:

$$0 \leq p^{\mathrm{D2}} \leq 50 \quad : \underline{\mu}^{\mathrm{D2}}, \overline{\mu}^{\mathrm{D2}}$$

Power balance equality:

$$p^{\mathrm{D1}} + p^{\mathrm{D2}} - p^{\mathrm{G1}} - p^{\mathrm{G2}} = 0 \quad : \lambda$$

Assume this unconstrained optimization problem is being solved by a fictitious player, the so-called "**price-setter**", who determines the market-clearing price by penalizing the power mismatch!

$$\underset{\lambda}{\text{Minimize}} \quad \lambda \left(p^{\mathrm{D1}} + p^{\mathrm{D2}} - p^{\mathrm{G1}} - p^{\mathrm{G2}}\right)$$

Optimization problem for each market player

For generator G1:

$$\underset{p^{\mathrm{G1}}}{\text{Maximize}} \quad p^{\mathrm{G1}}(\lambda - 12)$$

subject to:

$$0 \leq p^{\mathrm{G1}} \leq 100 \quad : \underline{\mu}^{\mathrm{G1}}, \overline{\mu}^{\mathrm{G1}}$$

For generator G2:

$$\underset{p^{\mathrm{G2}}}{\text{Maximize}} \quad p^{\mathrm{G2}}(\lambda - 20)$$

subject to:

$$0 \leq p^{\mathrm{G2}} \leq 80 \quad : \underline{\mu}^{\mathrm{G2}}, \overline{\mu}^{\mathrm{G2}}$$

For demand D1:

$$\underset{p^{\mathrm{D1}}}{\text{Maximize}} \quad p^{\mathrm{D1}}(40 - \lambda)$$

subject to:

$$0 \leq p^{\mathrm{D1}} \leq 100 \quad : \underline{\mu}^{\mathrm{D1}}, \overline{\mu}^{\mathrm{D1}}$$

For demand D2:

$$\underset{p^{\mathrm{D2}}}{\text{Maximize}} \quad p^{\mathrm{D2}}(35 - \lambda)$$

subject to:

$$0 \leq p^{\mathrm{D2}} \leq 50 \quad : \underline{\mu}^{\mathrm{D2}}, \overline{\mu}^{\mathrm{D2}}$$

Price-setter:

$$\underset{\lambda}{\text{Minimize}} \quad \lambda \left(p^{\mathrm{D1}} + p^{\mathrm{D2}} - p^{\mathrm{G1}} - p^{\mathrm{G2}}\right)$$

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$

subject to:

$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

For generator G2:

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$

subject to:

$$0 \leq p^{G2} \leq 80 \quad : \underline{\mu}^{G2}, \overline{\mu}^{G2}$$

For demand D1:

$$\underset{p^{D1}}{\text{Maximize}} \quad p^{D1}(40 - \lambda)$$

subject to:

$$0 \leq p^{D1} \leq 100 \quad : \underline{\mu}^{D1}, \overline{\mu}^{D1}$$

For demand D2:

$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$

subject to:

$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

Price-setter:

$$\underset{\lambda}{\text{Minimize}} \quad \lambda \left(p^{D1} + p^{D2} - p^{G1} - p^{G2}\right)$$

Question: Can we solve optimization problems above separately?

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$

subject to:

$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

For generator G2:

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$

subject to:

$$0 \leq p^{G2} \leq 80 \quad : \underline{\mu}^{G2}, \overline{\mu}^{G2}$$

For demand D1:

$$\underset{p^{D1}}{\text{Maximize}} \quad p^{D1}(40 - \lambda)$$

subject to:

$$0 \leq p^{D1} \leq 100 \quad : \underline{\mu}^{D1}, \overline{\mu}^{D1}$$

For demand D2:

$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$

subject to:

$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

Price-setter:

$$\underset{\lambda}{\text{Minimize}} \quad \lambda \left(p^{D1} + p^{D2} - p^{G1} - p^{G2}\right)$$

Question: Can we solve optimization problems above separately? No! Why?

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$
subject to:
$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

For generator G2:

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$
subject to:
$$0 \leq p^{G2} \leq 80 \quad : \underline{\mu}^{G2}, \overline{\mu}^{G2}$$

For demand D1:

$$\underset{p^{D1}}{\text{Maximize}} \quad p^{D1}(40 - \lambda)$$
subject to:
$$0 \leq p^{D1} \leq 100 \quad : \underline{\mu}^{D1}, \overline{\mu}^{D1}$$

For demand D2:

$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$
subject to:
$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

Price-setter:

$$\underset{\lambda}{\text{Minimize}} \quad \lambda \left(p^{D1} + p^{D2} - p^{G1} - p^{G2}\right)$$

- Market-clearing price is a variable for the price-setter, but a parameter for G1, G2, D1 and D2.
- Productions/consumptions are variables for G1, G2, D1 and D2, but parameters for the price-setter.

Optimization problem for each market player

For generator G1:

$$\underset{p^{\text{G1}}}{\text{Maximize}} \quad p^{\text{G1}}(\lambda - 12)$$

subject to:

$$0 \leq p^{\text{G1}} \leq 100 \quad : \underline{\mu}^{\text{G1}}, \overline{\mu}^{\text{G1}}$$

For generator G2:

$$\underset{p^{\text{G2}}}{\text{Maximize}} \quad p^{\text{G2}}(\lambda - 20)$$

subject to:

$$0 \leq p^{\text{G2}} \leq 80 \quad : \underline{\mu}^{\text{G2}}, \overline{\mu}^{\text{G2}}$$

For demand D1:

$$\underset{p^{\text{D1}}}{\text{Maximize}} \quad p^{\text{D1}}(40 - \lambda)$$

subject to:

$$0 \leq p^{\text{D1}} \leq 100 \quad : \underline{\mu}^{\text{D1}}, \overline{\mu}^{\text{D1}}$$

For demand D2:

$$\underset{p^{\text{D2}}}{\text{Maximize}} \quad p^{\text{D2}}(35 - \lambda)$$

subject to:

$$0 \leq p^{\text{D2}} \leq 50 \quad : \underline{\mu}^{\text{D2}}, \overline{\mu}^{\text{D2}}$$

Price-setter:

$$\underset{\lambda}{\text{Minimize}} \quad \lambda \left(p^{\text{D1}} + p^{\text{D2}} - p^{\text{G1}} - p^{\text{G2}}\right)$$

All five optimization problems above are linked, and should be solved all together!

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$

subject to:

$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

For generator G2:

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$

subject to:

$$0 \leq p^{G2} \leq 80 \quad : \underline{\mu}^{G2}, \overline{\mu}^{G2}$$

For demand D1:

$$\underset{p^{D1}}{\text{Maximize}} \quad p^{D1}(40 - \lambda)$$

subject to:

$$0 \leq p^{D1} \leq 100 \quad : \underline{\mu}^{D1}, \overline{\mu}^{D1}$$

For demand D2:

$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$

subject to:

$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

Price-setter:

$$\underset{\lambda}{\text{Minimize}} \quad \lambda \left(p^{D1} + p^{D2} - p^{G1} - p^{G2}\right)$$

This is a game-theoretic problem.

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$

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$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$

subject to:

$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

Price-setter:

$$\underset{\lambda}{\text{Minimize}} \quad \lambda \left(p^{D1} + p^{D2} - p^{G1} - p^{G2}\right)$$

This is a game-theoretic problem.

This specific problem is also known as "competitive equilibrium" problem!

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$

subject to:

$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

For generator G2:

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$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$

subject to:

$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

Price-setter:

$$\underset{\lambda}{\text{Minimize}} \quad \lambda \left(p^{D1} + p^{D2} - p^{G1} - p^{G2}\right)$$

Discussion:

What kind of game-theoretic problem is it? Is it a “**non-cooperative**” game? Or a “**cooperative**” one?

Some seminal works on competitive equilibrium

MATHEMATICAL METHODS OF ORGANIZING AND PLANNING PRODUCTION*†

L. V. KANTOROVICH

Leningrad State University

1939

Contents

L. V. Kantorovich, "Mathematical methods of organizing and planning production," *Management Science*, vol. 6, no. 4, pp. 366–422, 1960.

SPATIAL PRICE EQUILIBRIUM AND LINEAR PROGRAMMING

By PAUL A. SAMUELSON*

I.—*Introduction*

Increasingly, modern economic theorists are going beyond the formulation of equilibrium in terms of such marginal equalities as marginal revenue equal to marginal costs or wage rate equal to marginal value product. Instead they are reverting to an earlier and more fundamental aspect of a maximum position: namely, that from the top of a hill, whether or not it is locally flat, all movements are downward.

P. A. Samuelson, "Spatial price equilibrium and linear programming," *American Economic Review*, vol. 42, no. 3, pp. 283–303, 1952.

ECONOMETRICA

VOLUME 22 July, 1954 NUMBER 3

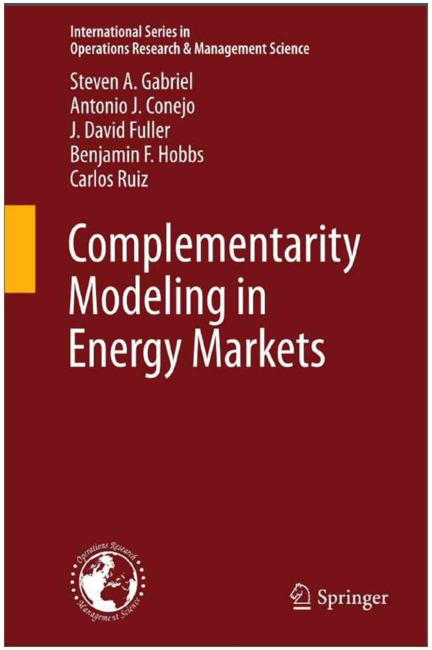
EXISTENCE OF AN EQUILIBRIUM FOR A COMPETITIVE ECONOMY

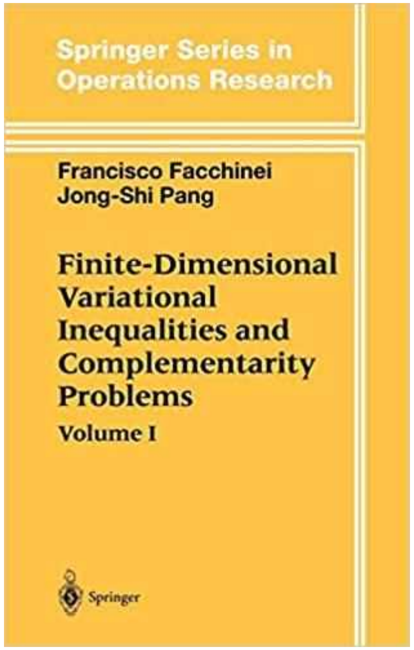
BY KENNETH J. ARROW AND GERARD DEBREU[1]

A. Wald has presented a model of production and a model of exchange and proofs of the existence of an equilibrium for each of them. Here proofs of the existence of an equilibrium are given for an *integrated* model of production, exchange and consumption. In addition the assumptions made on the technologies of producers and the tastes of consumers are significantly weaker than Wald's. Finally a simplification of the structure of the proofs has been made possible through use of the concept of an abstract economy, a generalization of that of a game.

K. J. Arrow and G. Debreu, "Existence of an equilibrium for a competitive economy," *Econometrica*, vol. 22, no. 3, pp. 265–290, 1954.

Relevant books and courses!





- Prof. Steven Gabriel's yearly short course at NTNU, "**Introduction Course in Complementarity Models and Equilibrium**": https://www.ntnu.edu/studies/courses/I%C3%988806#tab=omEmnet

- Prof. Uday Shanbhag's invited 5-day course at DTU in 2019. All video lectures are publicly available here: https://www.youtube.com/watch?v=PYXIzmxW53k&list=PLKLR7D59yU0fuZTH5wjgov31D3DXta_l-

Market clearing as an equilibrium problem

- If a solution to our equilibrium problem exists, it will be a "Nash equilibrium point", i.e.,

John Nash

No market participant can increase its profit by deviating unilaterally from the equilibrium solution!

Market clearing as an equilibrium problem

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Discussion:

- Is our market-clearing problem a "Nash equilibrium" (NE) problem?
- Or, is it a "generalized" Nash equilibrium (GNE)?
- What is the difference of NE and GNE? Which one is more appealing?

Market clearing as an equilibrium problem

- If a solution to our equilibrium problem exists, it will be a "Nash equilibrium point", i.e.,


John Nash

No market participant can increase its profit by deviating unilaterally from the equilibrium solution!

Recall the first question:
How to make sure all market participants are satisfied with the outcome of market-clearing optimization problem, and would not deviate from it?

Market clearing as an equilibrium problem

- If a solution to our equilibrium problem exists, it will be a "Nash equilibrium point", i.e.,


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No market participant can increase its profit by deviating unilaterally from the equilibrium solution!

Recall the first question:
How to make sure all market participants are satisfied with the outcome of market-clearing optimization problem, and would not deviate from it?

And new questions:
- Which problem should we solve to clear the market (optimization or equilibrium)?
- How to solve an equilibrium problem?

Market clearing: optimization vs equilibrium!

Equilibrium

For each generator:

Maximize profit

subject to production limits

For each demand:

Maximize utility

subject to consumption limits

Price-setter's problem

Optimization

Maximize market's social welfare

subject to:

- Production limits of generators
- Consumption limits of demands
- Power balance

How to solve the equilibrium problem?

Equilibrium

For each generator:

Maximize profit

subject to production limits

For each demand:

Maximize utility

subject to consumption limits

Price-setter's problem

How to solve the equilibrium problem?

Equilibrium

For each generator:

Maximize profit

subject to production limits

For each demand:

Maximize utility

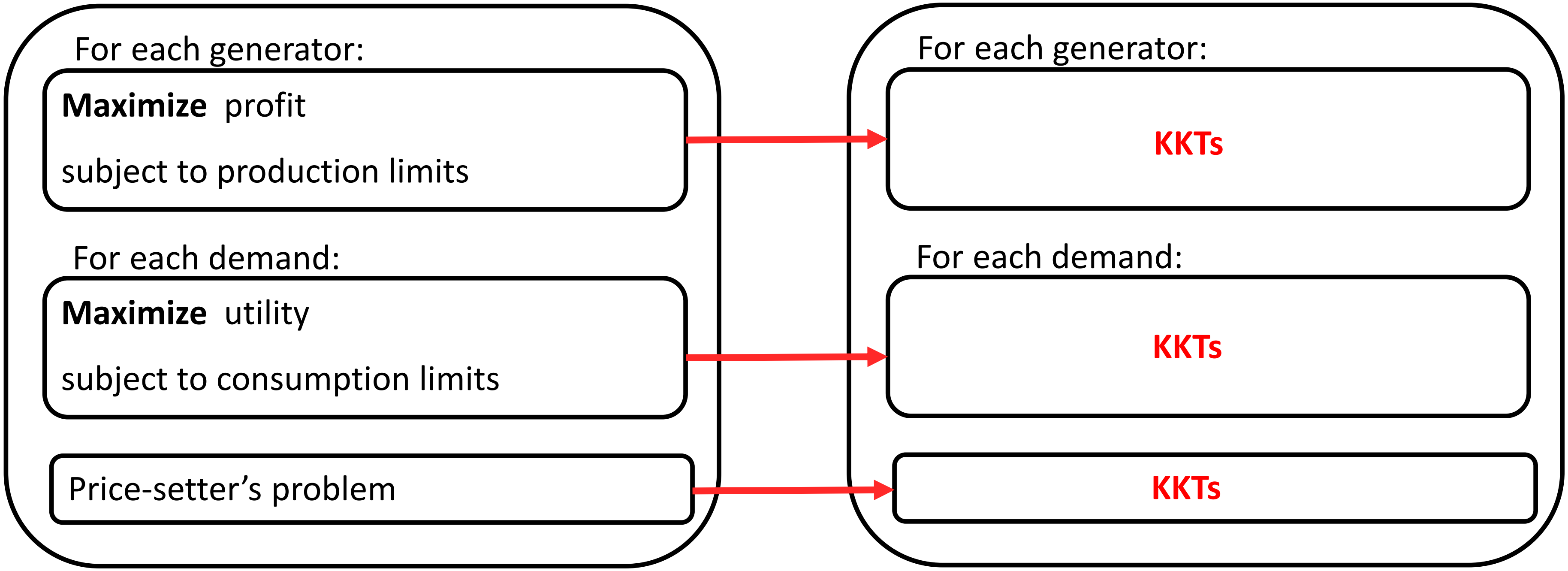
subject to consumption limits

Price-setter's problem

Replace each optimization problem within the equilibrium problem by its equivalent Karush-Kuhn-Tucker (KKT) optimality conditions! Recall that these conditions are a collection of equality and inequality conditions without any objective function!

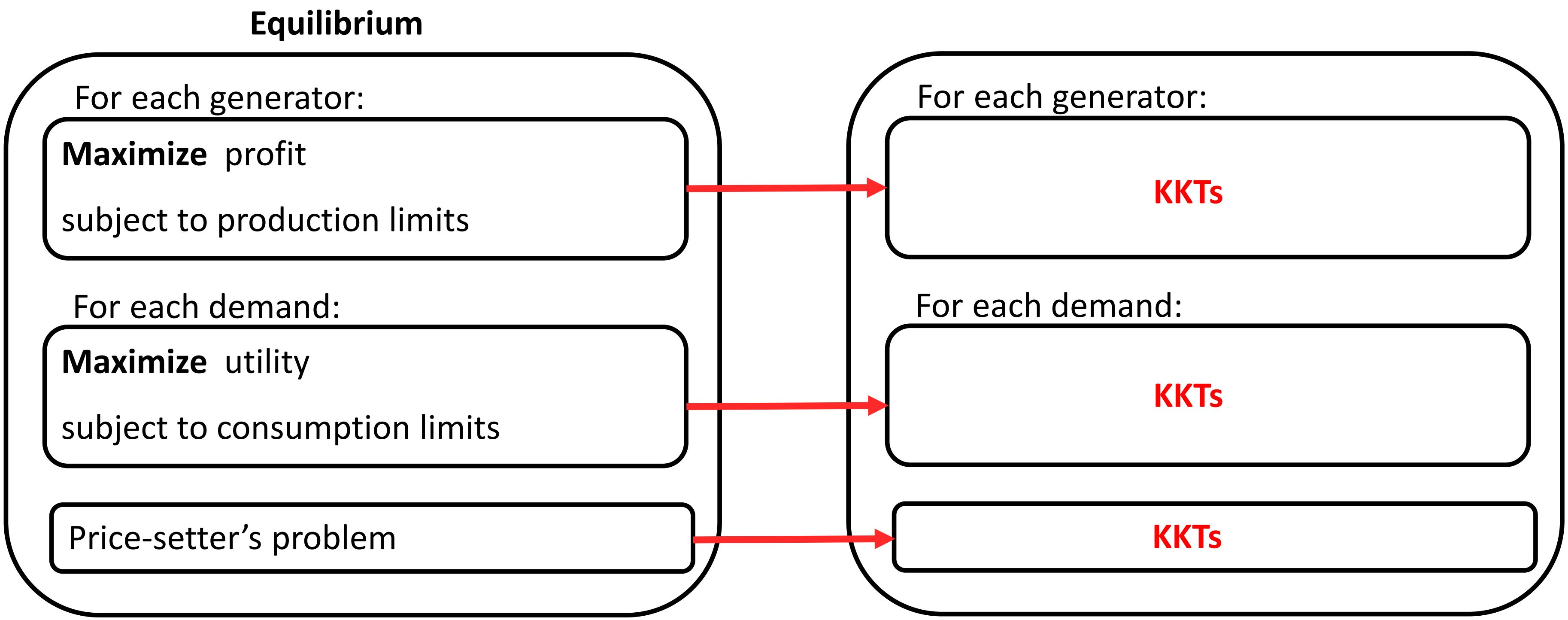
How to solve the equilibrium problem?

Equilibrium



Replace each optimization problem within the equilibrium problem by its equivalent Karush-Kuhn-Tucker (KKT) optimality conditions! Recall that these conditions are a collection of equality and inequality conditions without any objective function!

How to solve the equilibrium problem?



Mixed complementarity problem (MCP)

It is straightforward to solve!

Market clearing: optimization vs equilibrium!

Equilibrium

For each generator:

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Price-setter's problem

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Maximize market's social welfare

subject to:

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Let's check again the equilibrium and optimization problems above!

Market clearing: optimization vs equilibrium!

Equilibrium

For each generator:

Maximize profit

subject to production limits

For each demand:

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subject to consumption limits

Price-setter's problem

→ MCP

Optimization

Maximize market's social welfare

subject to:

- Production limits of generators
- Consumption limits of demands
- Power balance

→ KKTs

Market clearing: optimization vs equilibrium!

Equilibrium

For each generator:

Maximize profit

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Price-setter's problem

Optimization

Maximize market's social welfare

subject to:

- Production limits of generators
- Consumption limits of demands
- Power balance

MCP

KKTs

Do the MCP (obtained from equilibrium) and the KKTs (obtained from optimization) include **identical** conditions?

If so, the equilibrium and optimization problems above are "**equivalent**", i.e., any solution to the equilibrium problem is also a solution to the optimization problem and vice versa.

Market clearing as an equilibrium problem

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$

subject to:

$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$

subject to:

$$0 \leq p^{G2} \leq 80 \quad : \underline{\mu}^{G2}, \overline{\mu}^{G2}$$

$$\underset{p^{D1}}{\text{Maximize}} \quad p^{D1}(40 - \lambda)$$

subject to:

$$0 \leq p^{D1} \leq 100 \quad : \underline{\mu}^{D1}, \overline{\mu}^{D1}$$

$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$

subject to:

$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

$$\underset{\lambda}{\text{Minimize}} \quad \lambda \left(p^{D1} + p^{D2} - p^{G1} - p^{G2}\right)$$

Market clearing as an equilibrium problem

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$
subject to:
$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

KKTs →

$$12 - \lambda - \underline{\mu}^{G1} + \overline{\mu}^{G1} = 0$$
$$0 \leq p^{G1} \perp \underline{\mu}^{G1} \geq 0$$
$$0 \leq (100 - p^{G1}) \perp \overline{\mu}^{G1} \geq 0$$

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$
subject to:
$$0 \leq p^{G2} \leq 80 \quad : \underline{\mu}^{G2}, \overline{\mu}^{G2}$$

KKTs →

$$20 - \lambda - \underline{\mu}^{G2} + \overline{\mu}^{G2} = 0$$
$$0 \leq p^{G2} \perp \underline{\mu}^{G2} \geq 0$$
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→

$$p^{D1} + p^{D2} - p^{G1} - p^{G2} = 0$$

Market clearing as an equilibrium problem

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KKTs →

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subject to:
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$$0 \leq p^{D2} \perp \underline{\mu}^{D2} \geq 0$$
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$$\underset{\lambda}{\text{Minimize}} \quad \lambda\left(p^{D1} + p^{D2} - p^{G1} - p^{G2}\right)$$

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Mixed complementarity problem (MCP)

It is straightforward to solve!

Market clearing as an equilibrium problem

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$$p^{\mathrm{D1}} + p^{\mathrm{D2}} - p^{\mathrm{G1}} - p^{\mathrm{G2}} = 0$$

Mixed complementarity problem (MCP)

Market clearing as an equilibrium problem

Question:

Is this MCP identical to the KKTs of the market-clearing <u>optimization</u> problem? Let's check!

$$12 - \lambda - \underline{\mu}^{\mathrm{G1}} + \overline{\mu}^{\mathrm{G1}} = 0$$
$$0 \leq p^{\mathrm{G1}} \perp \underline{\mu}^{\mathrm{G1}} \geq 0$$
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$$\lambda - 40 - \underline{\mu}^{\mathrm{D1}} + \overline{\mu}^{\mathrm{D1}} = 0$$
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Mixed complementarity problem (MCP)

Market clearing as an equilibrium problem

Question:

Is this MCP identical to the KKTs of the market-clearing optimization problem? Let's check!

Answer: ?

Optimization

$$\underset{p^{G1},p^{G2},p^{D1},p^{D2}}{\text{Maximize}} \quad SW = [40p^{D1} + 35p^{D2}] - [12p^{G1} + 20p^{G2}] \tag{1a}$$

subject to:

$$0 \le p^{D1} \le 100 \quad : \underline{\mu}^{D1}, \overline{\mu}^{D1} \tag{1b}$$

$$0 \le p^{D2} \le 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2} \tag{1c}$$

$$0 \le p^{G1} \le 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1} \tag{1d}$$

$$0 \le p^{G2} \le 80 \quad : \underline{\mu}^{G2}, \overline{\mu}^{G2} \tag{1e}$$

$$p^{D1} + p^{D2} - p^{G1} - p^{G2} = 0 \quad : \lambda \tag{1f}$$

Mixed complementarity problem (MCP)

$$12 - \lambda - \underline{\mu}^{G1} + \overline{\mu}^{G1} = 0$$
$$0 \le p^{G1} \perp \underline{\mu}^{G1} \ge 0$$
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$$p^{D1} + p^{D2} - p^{G1} - p^{G2} = 0$$

Market clearing as an equilibrium problem

Question:

Is this MCP identical to the KKTs of the market-clearing <u>optimization</u> problem? Let's check!

Answer: Yes!

Optimization

$$\underset{p^{G1},p^{G2},p^{D1},p^{D2}}{\text{Maximize}} \quad SW = [40p^{D1} + 35p^{D2}] - [12p^{G1} + 20p^{G2}] \quad (1a)$$

subject to:

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$$0 \le p^{G1} \le 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1} \quad (1d)$$

$$0 \le p^{G2} \le 80 \quad : \underline{\mu}^{G2}, \overline{\mu}^{G2} \quad (1e)$$

$$p^{D1} + p^{D2} - p^{G1} - p^{G2} = 0 \quad : \lambda \quad (1f)$$

KKTs →

Mixed complementarity problem (MCP)

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$$0 \le (50 - p^{D2}) \perp \overline{\mu}^{D2} \ge 0$$

$$p^{D1} + p^{D2} - p^{G1} - p^{G2} = 0$$

Conclusions so far!

- The equilibrium and optimization forms of the market-clearing problem are equivalent, because their corresponding KKT conditions are identical!

- Both equilibrium and optimization forms of the market-clearing problem obtain the “Nash equilibrium solution”, i.e., no market player desires to deviate from the market-clearing outcomes!

Form 1: Market clearing as an optimization problem

$$\underset{p_g^{\mathrm{G}},\ p_d^{\mathrm{D}},\ \theta_n}{\text{Maximize}} \quad \sum_d U_d\ p_d^{\mathrm{D}} - \sum_g C_g\ p_g^{\mathrm{G}}$$

subject to:

$$0 \leq p_d^{\mathrm{D}} \leq \overline{P}_d^{\mathrm{D}} \quad : \underline{\mu}_d^{\mathrm{D}}, \overline{\mu}_d^{\mathrm{D}} \quad \forall d$$

$$0 \leq p_g^{\mathrm{G}} \leq \overline{P}_g^{\mathrm{G}} \quad : \underline{\mu}_g^{\mathrm{G}}, \overline{\mu}_g^{\mathrm{G}} \quad \forall g$$

$$\sum_{d \in \Psi_n} p_d^{\mathrm{D}} + \sum_{m \in \Omega_n} B_{n,m}(\theta_n - \theta_m) - \sum_{g \in \Psi_n} p_g^{\mathrm{G}} = 0 \quad : \lambda_n \quad \forall n$$

$$-F_{n,m} \leq B_{n,m}(\theta_n - \theta_m) \leq F_{n,m} \quad : \underline{\eta}_{n,m}, \overline{\eta}_{n,m} \quad \forall n, \forall m \in \Omega_n$$

$$\theta_{(n=ref)} = 0 \quad : \gamma$$

Form 2: Market clearing as an equilibrium problem

Each generator:

$$\underset{p_g^{\mathrm{G}}}{\text{Maximize}} \quad p_g^{\mathrm{G}}(\lambda_{n:g\in\Psi_n} - C_g)$$

subject to:

$$0 \le p_g^{\mathrm{G}} \le \overline{P}_g^{\mathrm{G}} \quad : \underline{\mu}_g^{\mathrm{G}}, \overline{\mu}_g^{\mathrm{G}}$$

Each elastic demand:

$$\underset{p_d^{\mathrm{D}}}{\text{Maximize}} \quad p_d^{\mathrm{D}}(U_d - \lambda_{n:d\in\Psi_n})$$

subject to:

$$0 \le p_d^{\mathrm{D}} \le \overline{P}_d^{\mathrm{D}} \quad : \underline{\mu}_d^{\mathrm{D}}, \overline{\mu}_d^{\mathrm{D}}$$

Transmission owner as a spatial arbitrager (it buys power at a bus and sells it back at another one):

$$\underset{\theta_n}{\text{Maximize}} \quad \sum_{n,(m\in\Omega_n)} \lambda_n[B_{n,m}(\theta_m - \theta_n)]$$

subject to:

$$-F_{n,m} \le B_{n,m}(\theta_n - \theta_m) \le F_{n,m} \quad : \underline{\eta}_{n,m}, \overline{\eta}_{n,m} \quad \forall n, \forall m \in \Omega_n$$

Price-setter:

$$\underset{\lambda_n}{\text{Minimize}} \quad \sum_n \lambda_n \left(\sum_{d\in\Psi_n} p_d^{\mathrm{D}} + \sum_{m\in\Omega_n} B_{n,m}(\theta_n - \theta_m) - \sum_{g\in\Psi_n} p_g^{\mathrm{G}} \right)$$

Form 2: Market clearing as an equilibrium problem

Each generator:

$$\underset{p_g^G}{\text{Maximize}} \quad p_g^G(\lambda_{n:g\in\Psi_n} - C_g)$$

subject to:

$$0 \le p_g^G \le \overline{P}_g^G \quad : \underline{\mu}_g^G, \overline{\mu}_g^G$$

Each elastic demand:

$$\underset{p_d^D}{\text{Maximize}} \quad p_d^D(U_d - \lambda_{n:d\in\Psi_n})$$

subject to:

$$0 \le p_d^D \le \overline{P}_d^D \quad : \underline{\mu}_d^D, \overline{\mu}_d^D$$

Transmission owner as a spatial arbitrager (it buys power at a bus and sells it back at another one):

$$\underset{\theta_n}{\text{Maximize}} \quad \sum_{n,(m\in\Omega_n)} \lambda_n[B_{n,m}(\theta_m - \theta_n)]$$

This term is known as "congestion rent"!

subject to:

$$-F_{n,m} \le B_{n,m}(\theta_n - \theta_m) \le F_{n,m} \quad : \underline{\eta}_{n,m}, \overline{\eta}_{n,m} \quad \forall n, \forall m \in \Omega_n$$

Price-setter:

$$\underset{\lambda_n}{\text{Minimize}} \quad \sum_n \lambda_n \left(\sum_{d\in\Psi_n} p_d^D + \sum_{m\in\Omega_n} B_{n,m}(\theta_n - \theta_m) - \sum_{g\in\Psi_n} p_g^G \right)$$

Closer look into congestion rent

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$$\underset{\theta_n}{\text{Maximize}} \quad \sum_{n,(m\in\Omega_n)} \lambda_n[B_{n,m}(\theta_m - \theta_n)]$$

Let's investigate if this objective function is correct!

subject to:

$$-F_{n,m} \leq B_{n,m}(\theta_n - \theta_m) \leq F_{n,m} \quad : \underline{\eta}_{n,m}, \overline{\eta}_{n,m} \quad \forall n, \forall m \in \Omega_n$$

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Question: What are “financial transmission rights (FTRs)”?

Form 2: Market clearing as an equilibrium problem

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subject to:

$$-F_{n,m} \leq B_{n,m}(\theta_n - \theta_m) \leq F_{n,m} \quad : \underline{\eta}_{n,m}, \overline{\eta}_{n,m} \quad \forall n, \forall m \in \Omega_n$$

Price-setter:

$$\underset{\lambda_n}{\text{Minimize}} \quad \sum_n \lambda_n \left(\sum_{d\in\Psi_n} p_d^{\mathrm{D}} + \sum_{m\in\Omega_n} B_{n,m}(\theta_n - \theta_m) - \sum_{g\in\Psi_n} p_g^{\mathrm{G}} \right)$$

Form 3: Market clearing as an MCP

$$-U_d + \lambda_{n:d\in\Psi_n} - \underline{\mu}_d^{\mathrm{D}} + \overline{\mu}_d^{\mathrm{D}} = 0 \quad \forall d$$

$$C_g - \lambda_{n:g\in\Psi_n} - \underline{\mu}_g^{\mathrm{G}} + \overline{\mu}_g^{\mathrm{G}} = 0 \quad \forall g$$

$$\sum_{m\in\Omega_n} B_{n,m}(\lambda_n - \lambda_m + \overline{\eta}_{n,m} - \overline{\eta}_{m,n} - \underline{\eta}_{n,m} + \underline{\eta}_{m,n}) + (\gamma)_{n=ref} = 0 \quad \forall n$$

$$\sum_{d\in\Psi_n} p_d^{\mathrm{D}} + \sum_{m\in\Omega_n} B_{n,m}(\theta_n - \theta_m) - \sum_{g\in\Psi_n} p_g^{\mathrm{G}} = 0 \quad \forall n$$

$$\theta_{(n=ref)} = 0$$

$$0 \leq p_g^{\mathrm{G}} \perp \underline{\mu}_g^{\mathrm{G}} \geq 0 \quad \forall g$$

$$0 \leq [\overline{P}_g^{\mathrm{G}} - p_g^{\mathrm{G}}] \perp \overline{\mu}_g^{\mathrm{G}} \geq 0 \quad \forall g$$

$$0 \leq p_d^{\mathrm{D}} \perp \underline{\mu}_d^{\mathrm{D}} \geq 0 \quad \forall d$$

$$0 \leq [\overline{P}_d^{\mathrm{D}} - p_d^{\mathrm{D}}] \perp \overline{\mu}_d^{\mathrm{D}} \geq 0 \quad \forall d$$

$$0 \leq [F_{n,m} + B_{n,m}(\theta_n - \theta_m)] \perp \underline{\eta}_{n,m} \geq 0 \quad \forall n, \forall m \in \Omega_n$$

$$0 \leq [F_{n,m} - B_{n,m}(\theta_n - \theta_m)] \perp \overline{\eta}_{n,m} \geq 0 \quad \forall n, \forall m \in \Omega_n$$

Conclusion

All three forms of the market-clearing problem, i.e.,

- optimization
- equilibrium
- MCP

are equivalent!

Exercises

1- Which form of the market-clearing problem (optimization or equilibrium) is more appealing to market operators?

2- Consider an equilibrium form of the market-clearing problem. Is it possible to solve it iteratively without deriving KKTs? If so, what are the pros and cons?

Guide: Consider an iterative mechanism, in which the market operator fixes a set of initial prices, and then each market participant makes her own dispatch decisions accordingly. Based on the participants' dispatch decisions, the market operator checks whether nodal power balance conditions hold or not. If not, the operator "systematically" adjusts those prices and disseminates the updated prices among participants. This can be continued until there is no demand-supply mismatch. If interested, read about "Walrasian auction" and its "tâtonnement process", which indeed requires a decomposition technique (e.g., Lagrangian relaxation or ADMM).

3- For a given Nash equilibrium (NE) problem, how to mathematically identify that an equivalent optimization problem exists?

Guide: check chapter 4 of the book by S. Gabriel et al. (available on DTU Inside) and learn about "Principle of Symmetry", referring to the symmetry of Jacobian matrix. Search how we can derive the Jacobian matrix of a game. You can also check Theorem 1.3.1 of the book by F. Facchinei and J.-S. Pang (available on DTU Inside).

4- Investigate how the solution existence and the solution uniqueness for a Nash equilibrium (NE) can be mathematically proven.

Guide: For uniqueness, search about "monotonicity" property of a game. How can we ensure a game is strongly monotone by checking the Jacobian matrix? You can also read about "degree theory".

Thanks for your attention!

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Lecture 3: Desirable market properties

Jalal Kazempour

The 6th KIOS Graduate Training School

September 3, 2024

Recap

- Optimization form of the market-clearing problem maximizes market's social welfare.

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- Therefore, the market's social welfare is maximized, while each market player obtains its maximum objective.

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- Optimization and equilibrium forms of the market-clearing problem are "equivalent", since their corresponding KKT conditions are identical.

- Therefore, the market's social welfare is maximized, while each market player obtains its maximum objective.

- This means that the market-clearing mechanism is **efficient**.

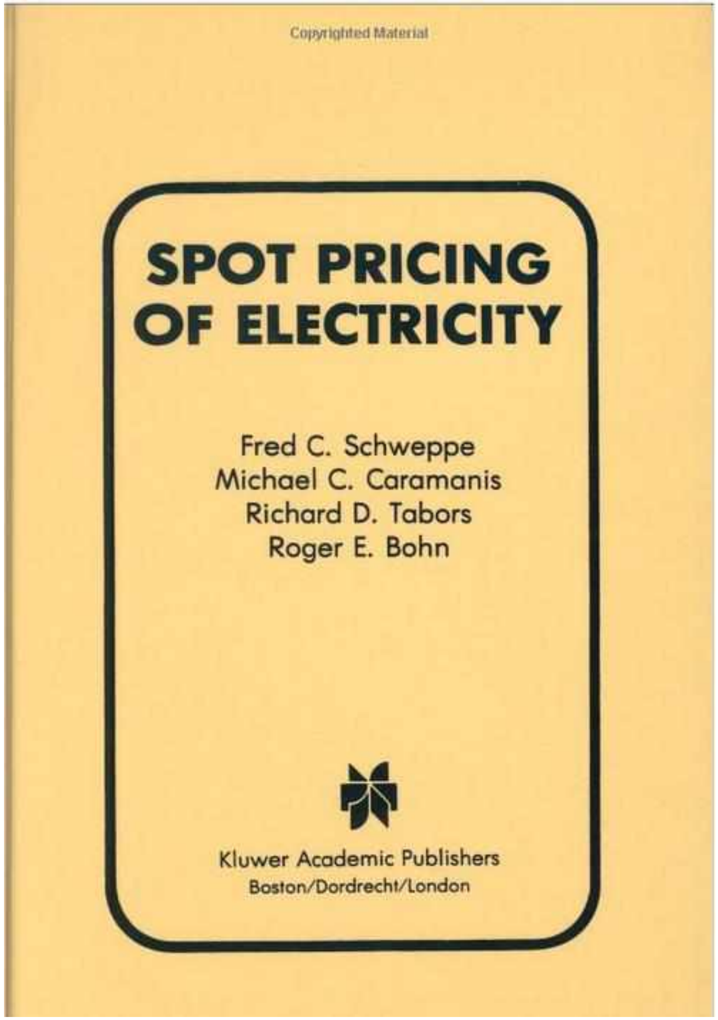
- A market fails (market failure) if for any reason, social welfare is not maximized, or there is any player who desires to unilaterally deviate from market-clearing outcomes.

Discussion

Question:

In addition to achieving market efficiency, is there any other desirable economic property for a market-clearing mechanism?

A relevant and very nice book!



Desirable economic properties

<u>Four</u> desirable properties of market-clearing mechanisms are:

Market efficiency	**Incentive compatibility**
Cost recovery	**Revenue adequacy**

Desirable economic properties

Four desirable properties of market-clearing mechanisms are:

Market efficiency	Incentive compatibility
Cost recovery	Revenue adequacy

In an efficient market, the social welfare is maximized, and no one desires to unilaterally deviate from the market outcomes.

Desirable economic properties

Four desirable properties of market-clearing mechanisms are:

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In an incentive-compatible market:

- Every market player can maximize its objective just by acting according to her "true" preferences.

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- Every market player can maximize its objective just by acting according to her "true" preferences.
- In an incentive-compatible market, if the production cost of a generator is \$12/MWh, the **dominant** (most profitable) strategy for that generator is to offer "trustfully" at \$12/MWh, not at any different price!

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- In other words, no market player desires to exercise "market power" by behaving "strategically", i.e., by submitting "strategic" offers.

Desirable economic properties

<u>Four</u> desirable properties of market-clearing mechanisms are:

Market efficiency	**Incentive compatibility**
Cost recovery	**Revenue adequacy**

- Cost recovery refers to a condition under which every market player is able to recover her operational (but not necessarily capital) cost. In other words, her operational profit is always <u>non-negative</u>.

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- Cost recovery refers to a condition under which every market player is able to recover her operational (but not necessarily capital) cost. In other words, her operational profit is always non-negative.

- This property is also known as "**individual rationality**" (although based on some definition in the literature there might be slight differences).

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Four desirable properties of market-clearing mechanisms are:

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- Revenue adequacy refers to a condition under which the market operator never incurs a financial deficit.

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- Revenue adequacy refers to a condition under which the market operator never incurs a financial deficit.
- In other words, the total payment that the market operator receives from demands is always higher than or equal to her total payment to generators, curtailed loads, transmission operator, etc.
- As a specific status of revenue adequacy, the market is "**budget balance**" if the market operator has neither financial deficit nor excess.

Discussion

Question:
Is there any market-clearing mechanism ensuring all four properties?

Discussion

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Is there any market-clearing mechanism ensuring all four properties?

Answer:
No!

- Based on Hurwicz theorem (also known as "impossibility theorem") [1]-[2], no mechanism is capable of achieving all those properties at the same time!

- We have to find a "trade-off" among properties achieved and those lost.

[1] L. Hurwicz, ``On Informationally Decentralized Systems'' in *Decision and Organization*, edited by C.B. McGuire and R. Radner, Amsterdam, 1972.

[2] R. Myerson and M. A. Satterthwaite, "Efficient mechanisms for bilateral trading," *Journal of Economic Theory*, vol. 28, pp. 265–281, 1983.

Discussion

Recall that market-clearing models determine the nodal market-clearing prices (LMPs) based on dual variable of nodal power balance equalities. Let’s refer this pricing method to as “LMP-based market mechanism”.

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Equilibrium

For each generator:

Maximize profit

subject to production limits

For each demand:

Maximize utility

subject to consumption limits

Power balance

Optimization

Maximize market's social welfare

subject to:

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Question:

Which properties are ensured in the LMP-based market mechanism?

Properties of LMP-based market mechanism

“Incentive compatibility” ensured?

“Market efficiency” ensured?

“Revenue adequacy” ensured?

“Cost recovery” ensured?

Properties of LMP-based market mechanism

“Incentive compatibility” ensured?

No! A market player (the so-called “strategic” player) may exercise “market power” by not trustfully offering in terms of price and/or quantity! ☹

“Market efficiency” ensured?

“Revenue adequacy” ensured?

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"Incentive compatibility" ensured?

No! A market player (the so-called "strategic" player) may exercise "market power" by not trustfully offering in terms of price and/or quantity! ☹

"Market efficiency" ensured?

No! in the sense that if "market power" is exercised, the market's social welfare will be decreased. ☹

"Revenue adequacy" ensured?

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No! A market player (the so-called "strategic" player) may exercise "market power" by not trustfully offering in terms of price and/or quantity!

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No! in the sense that if "market power" is exercised, the market's social welfare will be decreased.

"Revenue adequacy" ensured?

Yes! Proof as an exercise; see the next slide!

"Cost recovery" ensured?

Yes! Proof as an exercise; see the next slide!

Exercise 1

Provide a mathematical proof that the LMP-based market mechanism ensures "revenue adequacy" and even "budget balance".

Guide:

Step 1- Consider the nodal power balance equality for bus *n*.

Step 2- Multiply each term within the equality of Step 1 by the LMP at that bus.

Step 3- Consider the summation of all equalities obtained in Step 2 for all buses. What does the resulting equality mean?

Exercise 2

Provide a mathematical proof that the LMP-based market mechanism ensures "cost recovery" for all market players.

Guide:

Step 1- Consider the equilibrium form of the market-clearing problem.

Step 2- For each generator's optimization problem, derive the corresponding "strong duality" condition, which enforces the equality of objective function of primal and dual problems at the optimal point. The primal objective function is generator's profit. Check the terms within the dual objective function -- are they all non-negative? If so, what does it mean?

Step 3- Similar to Step 2, investigate the cost recovery for elastic demands and transmission system operator using the equilibrium problem in Step 1.

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Step 3- Similar to Step 2, investigate the cost recovery for elastic demands and transmission system operator using the equilibrium problem in Step 1.

Question: If the lower bound for the generation level of a generator is a positive (non-zero) value, can we still ensure cost recovery?

Unit commitment

Recall that we have ignored unit commitment (UC) constraints of thermal generators within the market-clearing problem.

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Recall that we have ignored unit commitment (UC) constraints of thermal generators within the market-clearing problem. These UC constraints include:

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- etc.

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Some references explaining the formulation of UC constraints:

[1] M. Carrión and J. M. Arroyo, "A computationally efficient mixed-integer linear formulation for the thermal unit commitment problem," *IEEE Transactions on power systems*, vol. 21, no. 3, pp. 1371-1378, 2006.

[2] J. Ostrowski, M. F. Anjos, A. Vannelli, "Tight mixed integer linear programming formulations for the unit commitment problem," *IEEE Transactions on Power Systems,* vol. 27, no. 1, pp. 39-46, 2012.

[3] M. F. Anjos and A. J. Conejo, "Unit commitment in electric energy systems," *Foundations and Trends® in Electric Energy Systems*, vol. 1 no. 4, pp. 220-310, 2017.

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- Reference [1] uses one type of binaries only (on/off status of generators) to model all UC constraints.
- Reference [2] models the same UC constraints as in [1], but using three types of binaries (on/off, start-up, and shut-down status of generators).
- Both models obtain the same results, but the model in [2] is computationally faster than the one in [1]! Any idea why?

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- etc.

These constraints require binary variables!

One more reference!

If you would like to "relax" binary variables of UC constraints in a "tight" way, e.g., for capacity expansion planning studies:

[4] B. Hua, R. Baldick, and J. Wang, "Representing operational flexibility in generation expansion planning through convex relaxation of unit commitment," *IEEE Transactions on power systems,* vol. 33, no. 2, pp. 2272–2281, 2018.

Discussion

Note: The unit commitment problem is a mixed-integer linear problem (MILP), and we still need dual variables to derive market prices!

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Question: How to derive dual variables in a MILP? Is there any mathematical challenge?

Some seminal papers about duality theory for integer programming:

[1] E.L. Johnson, "Cyclic groups, cutting planes and shortest paths", *Mathematical programming*, 1973.

[2] L. A. Wolsey, "Integer programming duality: Price functions and sensitivity analysis," Mathematical Programming, vol. 20, no. 1, pp. 173-195, 1981.

[3] A. C. Williams, "Marginal values in mixed integer linear programming," Mathematical Programming, vol. 44, no. 1-3, pp. 67-75, 1989.

Discussion

Questions: When UC constraints are included,

1- How to derive "market prices"?

2- Does the LMP-based market-clearing mechanism still guarantee achieving "cost recovery" for all market players?

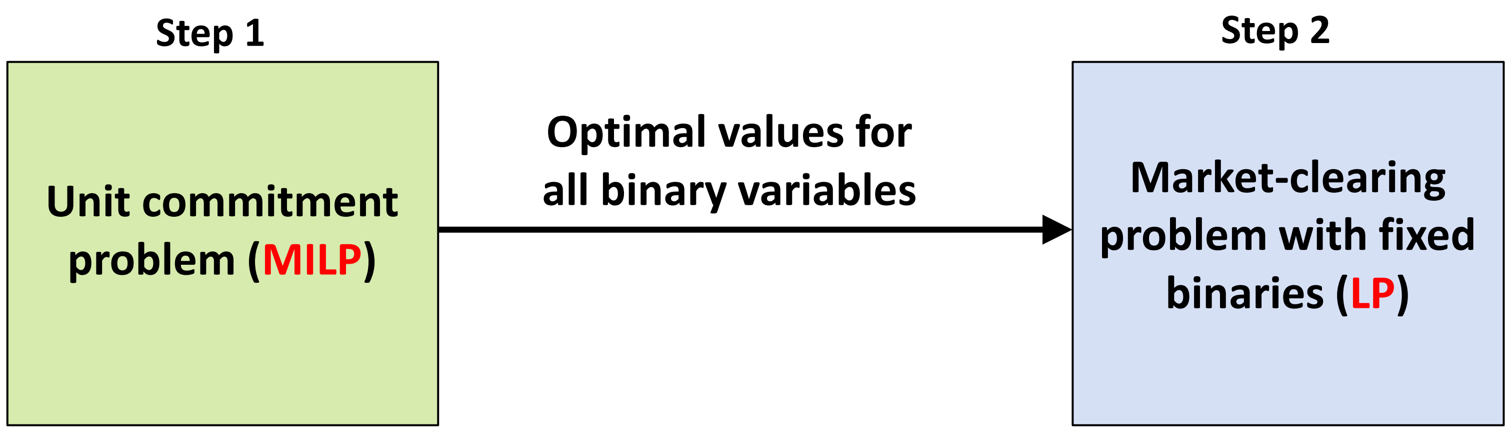
Pricing with unit commitment constraints

Step 1

Unit commitment problem (MILP)

- **Step 1:** Solve the unit commitment problem, and obtain the optimal values for all binary variables

Pricing with unit commitment constraints



- **Step 1:** Solve the unit commitment problem, and obtain the optimal values for all binary variables

- **Step 2:** Solve the same problem while binary variables are replaced by their optimal values (0 or 1) obtained in Step 1. This results in a linear problem (LP), so determine the LMPs based on dual variables.

Step 1: unit commitment problem

Step 1: unit commitment problem

Let’s add the following UC constraints to the market-clearing problem:

- The minimum production level of generators,
- Start-up cost of generators.

Step 1: unit commitment problem

$$\underset{p^{\mathrm{G}}_{g,t}, u^{\mathrm{G}}_{g,t}, s^{\mathrm{G}}_{g,t},\ p^{\mathrm{D}}_{d,t},\ \theta_{n,t}}{\text{Maximize}} \quad \sum_{d,t} U_{d,t}\ p^{\mathrm{D}}_{d,t} - \sum_{g,t} [s^{\mathrm{G}}_{g,t} + (C_g\ p^{\mathrm{G}}_{g,t})]$$

subject to:

$$0 \leq p^{\mathrm{D}}_{d,t} \leq \overline{P}^{\mathrm{D}}_{d,t} \quad \forall d, \forall t$$

$$\underline{P}^{\mathrm{G}}_{g,t}\ u^{\mathrm{G}}_{g,t} \leq p^{\mathrm{G}}_{g,t} \leq \overline{P}^{\mathrm{G}}_{g}\ u^{\mathrm{G}}_{g,t} \quad \forall g, \forall t$$

$$\sum_{d \in \Psi_n} p^{\mathrm{D}}_{d,t} + \sum_{m \in \Omega_n} B_{n,m}(\theta_{n,t} - \theta_{m,t}) - \sum_{g \in \Psi_n} p^{\mathrm{G}}_{g,t} = 0 \quad \forall n, \forall t$$

$$-F_{n,m} \leq B_{n,m}(\theta_{n,t} - \theta_{m,t}) \leq F_{n,m} \quad \forall n, \forall m \in \Omega_n, \forall t$$

$$\theta_{(n=ref),t} = 0 \quad \forall t$$

$$0 \leq s^{\mathrm{G}}_{g,t} \geq C^{\mathrm{SU}}_{g}(u^{\mathrm{G}}_{g,t} - u^{\mathrm{G}}_{g,t-1}) \quad \forall g, \forall t$$

$$u^{\mathrm{G}}_{g,t} \in \{0, 1\} \quad \forall g, \forall t$$

Step 1: unit commitment problem

$$\underset{p^{\mathrm{G}}_{g,t}, u^{\mathrm{G}}_{g,t}, s^{\mathrm{G}}_{g,t}, p^{\mathrm{D}}_{d,t}, \theta_{n,t}}{\text{Maximize}} \quad \sum_{d,t} U_{d,t}\, p^{\mathrm{D}}_{d,t} - \sum_{g,t} [s^{\mathrm{G}}_{g,t} + (C_g\, p^{\mathrm{G}}_{g,t})]$$

subject to:

$$0 \leq p^{\mathrm{D}}_{d,t} \leq \overline{P}^{\mathrm{D}}_{d,t} \quad \forall d, \forall t$$

Index for time periods

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$$u^{\mathrm{G}}_{g,t} \in \{0,1\} \quad \forall g, \forall t$$

Step 1: unit commitment problem

$$\underset{p^{\text{G}}_{g,t}, u^{\text{G}}_{g,t}, s^{\text{G}}_{g,t},\, p^{\text{D}}_{d,t},\, \theta_{n,t}}{\text{Maximize}} \quad \sum_{d,t} U_{d,t}\, p^{\text{D}}_{d,t} - \sum_{g,t} [s^{\text{G}}_{g,t} + (C_g\, p^{\text{G}}_{g,t})]$$

subject to:

$$0 \leq p^{\text{D}}_{d,t} \leq \overline{P}^{\text{D}}_{d,t} \quad \forall d, \forall t$$

Index for time periods

$$\underline{P}^{\text{G}}_{g,t}\, u^{\text{G}}_{g,t} \leq p^{\text{G}}_{g,t} \leq \overline{P}^{\text{G}}_{g}\, u^{\text{G}}_{g,t} \quad \forall g, \forall t$$

$$\sum_{d \in \Psi_n} p^{\text{D}}_{d,t} + \sum_{m \in \Omega_n} B_{n,m}(\theta_{n,t} - \theta_{m,t}) - \sum_{g \in \Psi_n} p^{\text{G}}_{g,t} = 0 \quad \forall n, \forall t$$

$$-F_{n,m} \leq B_{n,m}(\theta_{n,t} - \theta_{m,t}) \leq F_{n,m} \quad \forall n, \forall m \in \Omega_n, \forall t$$

$$\theta_{(n=ref),t} = 0 \quad \forall t$$

$$0 \leq s^{\text{G}}_{g,t} \geq C^{\text{SU}}_{g}(u^{\text{G}}_{g,t} - u^{\text{G}}_{g,t-1}) \quad \forall g, \forall t$$

Binary variables

$$u^{\text{G}}_{g,t} \in \{0, 1\} \quad \forall g, \forall t$$

Step 1: unit commitment problem

$$\underset{p^{G}_{g,t}, u^{G}_{g,t}, s^{G}_{g,t},\ p^{D}_{d,t},\ \theta_{n,t}}{\text{Maximize}} \quad \sum_{d,t} U_{d,t}\ p^{D}_{d,t} - \sum_{g,t}[s^{G}_{g,t} + (C_g\ p^{G}_{g,t})]$$

subject to:

$$0 \leq p^{D}_{d,t} \leq \overline{P}^{D}_{d,t} \quad \forall d, \forall t$$

Index for time periods

Binary variable appears in the lower and upper bounds of generation level

$$\underline{P}^{G}_{g,t}\ u^{G}_{g,t} \leq p^{G}_{g,t} \leq \overline{P}^{G}_{g}\ u^{G}_{g,t} \quad \forall g, \forall t$$

$$\sum_{d \in \Psi_n} p^{D}_{d,t} + \sum_{m \in \Omega_n} B_{n,m}(\theta_{n,t} - \theta_{m,t}) - \sum_{g \in \Psi_n} p^{G}_{g,t} = 0 \quad \forall n, \forall t$$

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$$\theta_{(n=ref),t} = 0 \quad \forall t$$

$$0 \leq s^{G}_{g,t} \geq C^{SU}_{g}(u^{G}_{g,t} - u^{G}_{g,t-1}) \quad \forall g, \forall t$$

Binary variables

$$u^{G}_{g,t} \in \{0, 1\} \quad \forall g, \forall t$$

Step 1: unit commitment problem

$$\underset{p^G_{g,t}, u^G_{g,t}, s^G_{g,t}, p^D_{d,t}, \theta_{n,t}}{\text{Maximize}} \quad \sum_{d,t} U_{d,t}\, p^D_{d,t} - \sum_{g,t} [s^G_{g,t} + (C_g\, p^G_{g,t})]$$

Variable: start-up cost of generator *g* in time *t*

subject to:

$$0 \leq p^D_{d,t} \leq \overline{P}^D_{d,t} \quad \forall d, \forall t$$

Index for time periods

Binary variable appears in the lower and upper bounds of generation level

$$\underline{P}^G_{g,t}\, u^G_{g,t} \leq p^G_{g,t} \leq \overline{P}^G_g\, u^G_{g,t} \quad \forall g, \forall t$$

$$\sum_{d \in \Psi_n} p^D_{d,t} + \sum_{m \in \Omega_n} B_{n,m}(\theta_{n,t} - \theta_{m,t}) - \sum_{g \in \Psi_n} p^G_{g,t} = 0 \quad \forall n, \forall t$$

$$-F_{n,m} \leq B_{n,m}(\theta_{n,t} - \theta_{m,t}) \leq F_{n,m} \quad \forall n, \forall m \in \Omega_n, \forall t$$

$$\theta_{(n=ref),t} = 0 \quad \forall t$$

$$0 \leq s^G_{g,t} \geq C^{SU}_g (u^G_{g,t} - u^G_{g,t-1}) \quad \forall g, \forall t$$

Binary variables

$$u^G_{g,t} \in \{0, 1\} \quad \forall g, \forall t$$

Step 1: unit commitment problem

$$\underset{p^{G}_{g,t},u^{G}_{g,t},s^{G}_{g,t},\ p^{D}_{d,t},\ \theta_{n,t}}{\text{Maximize}} \quad \sum_{d,t} U_{d,t}\ p^{D}_{d,t} - \sum_{g,t} [s^{G}_{g,t} + (C_g\ p^{G}_{g,t})]$$

Variable: start-up cost of generator *g* in time *t*

subject to:

$$0 \leq p^{D}_{d,t} \leq \overline{P}^{D}_{d,t} \quad \forall d, \forall t$$

Index for time periods

Binary variable appears in the lower and upper bounds of generation level

$$\underline{P}^{G}_{g,t}\ u^{G}_{g,t} \leq p^{G}_{g,t} \leq \overline{P}^{G}_{g}\ u^{G}_{g,t} \quad \forall g, \forall t$$

$$\sum_{d\in\Psi_n} p^{D}_{d,t} + \sum_{m\in\Omega_n} B_{n,m}(\theta_{n,t} - \theta_{m,t}) - \sum_{g\in\Psi_n} p^{G}_{g,t} = 0 \quad \forall n, \forall t$$

$$-F_{n,m} \leq B_{n,m}(\theta_{n,t} - \theta_{m,t}) \leq F_{n,m} \quad \forall n, \forall m \in \Omega_n, \forall t$$

$$\theta_{(n=ref),t} = 0 \quad \forall t$$

$$0 \leq s^{G}_{g,t} \geq C^{SU}_{g}(u^{G}_{g,t} - u^{G}_{g,t-1}) \quad \forall g, \forall t$$

Start-up cost constraints

Binary variables

$$u^{G}_{g,t} \in \{0,1\} \quad \forall g, \forall t$$

Step 1: unit commitment problem

$$\underset{p^{G}_{g,t}, u^{G}_{g,t}, s^{G}_{g,t},\ p^{D}_{d,t},\ \theta_{n,t}}{\text{Maximize}} \quad \sum_{d,t} U_{d,t}\ p^{D}_{d,t} - \sum_{g,t} [s^{G}_{g,t} + (C_g\ p^{G}_{g,t})]$$

Variable: start-up cost of generator *g* in time *t*

subject to:

$$0 \leq p^{D}_{d,t} \leq \overline{P}^{D}_{d,t} \quad \forall d, \forall t$$

Index for time periods

Binary variable appears in the lower and upper bounds of generation level

$$\underline{P}^{G}_{g,t}\ u^{G}_{g,t} \leq p^{G}_{g,t} \leq \overline{P}^{G}_{g}\ u^{G}_{g,t} \quad \forall g, \forall t$$

$$\sum_{d \in \Psi_n} p^{D}_{d,t} + \sum_{m \in \Omega_n} B_{n,m}(\theta_{n,t} - \theta_{m,t}) - \sum_{g \in \Psi_n} p^{G}_{g,t} = 0 \quad \forall n, \forall t$$

$$-F_{n,m} \leq B_{n,m}(\theta_{n,t} - \theta_{m,t}) \leq F_{n,m} \quad \forall n, \forall m \in \Omega_n, \forall t$$

Parameter: start-up cost of generator *g*

$$\theta_{(n=ref),t} = 0 \quad \forall t$$

$$0 \leq s^{G}_{g,t} \geq C^{SU}_{g}(u^{G}_{g,t} - u^{G}_{g,t-1}) \quad \forall g, \forall t$$

Start-up cost constraints

Binary variables

$$u^{G}_{g,t} \in \{0,1\} \quad \forall g, \forall t$$

Step 2: LP with fixed binaries

Step 2: LP with fixed binaries

$$\underset{p^{\mathrm{G}}_{g,t}, s^{\mathrm{G}}_{g,t},\ p^{\mathrm{D}}_{d,t},\ \theta_{n,t}}{\text{Maximize}} \quad \sum_{d,t} U_{d,t}\ p^{\mathrm{D}}_{d,t} - \sum_{g,t}[s^{\mathrm{G}}_{g,t} + (C_g\ p^{\mathrm{G}}_{g,t})]$$

subject to:

$$0 \le p^{\mathrm{D}}_{d,t} \le \overline{P}^{\mathrm{D}}_{d,t} \quad \forall d, \forall t$$

$$\underline{P}^{\mathrm{G}}_{g,t}\ u^{\mathrm{G,fixed}}_{g,t} \le p^{\mathrm{G}}_{g,t} \le \overline{P}^{\mathrm{G}}_{g}\ u^{\mathrm{G,fixed}}_{g,t} \quad \forall g, \forall t$$

$$\sum_{d\in\Psi_n} p^{\mathrm{D}}_{d,t} + \sum_{m\in\Omega_n} B_{n,m}(\theta_{n,t} - \theta_{m,t}) - \sum_{g\in\Psi_n} p^{\mathrm{G}}_{g,t} = 0 \qquad : \lambda_{n,t} \quad \forall n, \forall t$$

$$-F_{n,m} \le B_{n,m}(\theta_{n,t} - \theta_{m,t}) \le F_{n,m} \quad \forall n, \forall m \in \Omega_n, \forall t$$

$$\theta_{(n=ref),t} = 0 \quad \forall t$$

$$0 \le s^{\mathrm{G}}_{g,t} \ge C^{\mathrm{SU}}_{g}(u^{\mathrm{G,fixed}}_{g,t} - u^{\mathrm{G,fixed}}_{g,t-1}) \quad \forall g, \forall t$$

Step 2: LP with fixed binaries

$$\underset{p^{G}_{g,t},\, s^{G}_{g,t},\, p^{D}_{d,t},\, \theta_{n,t}}{\text{Maximize}} \quad \sum_{d,t} U_{d,t}\, p^{D}_{d,t} - \sum_{g,t} [s^{G}_{g,t} + (C_g\, p^{G}_{g,t})]$$

subject to:

$$0 \leq p^{D}_{d,t} \leq \overline{P}^{D}_{d,t} \quad \forall d, \forall t$$

$$\underline{P}^{G}_{g,t}\, u^{G,\text{fixed}}_{g,t} \leq p^{G}_{g,t} \leq \overline{P}^{G}_{g}\, u^{G,\text{fixed}}_{g,t} \quad \forall g, \forall t$$

$$\sum_{d \in \Psi_n} p^{D}_{d,t} + \sum_{m \in \Omega_n} B_{n,m}(\theta_{n,t} - \theta_{m,t}) - \sum_{g \in \Psi_n} p^{G}_{g,t} = 0 \quad : \lambda_{n,t} \quad \forall n, \forall t$$

$$-F_{n,m} \leq B_{n,m}(\theta_{n,t} - \theta_{m,t}) \leq F_{n,m} \quad \forall n, \forall m \in \Omega_n, \forall t$$

$$\theta_{(n=ref),t} = 0 \quad \forall t$$

$$0 \leq s^{G}_{g,t} \geq C^{SU}_{g}(u^{G,\text{fixed}}_{g,t} - u^{G,\text{fixed}}_{g,t-1}) \quad \forall g, \forall t$$

Questions:
Does the start-up cost affect market prices?

Step 2: LP with fixed binaries

$$\underset{p^{\mathrm{G}}_{g,t},s^{\mathrm{G}}_{g,t},\ p^{\mathrm{D}}_{d,t},\ \theta_{n,t}}{\text{Maximize}} \quad \sum_{d,t} U_{d,t}\ p^{\mathrm{D}}_{d,t} - \sum_{g,t} [s^{\mathrm{G}}_{g,t} + (C_g\ p^{\mathrm{G}}_{g,t})]$$

subject to:

$$0 \leq p^{\mathrm{D}}_{d,t} \leq \overline{P}^{\mathrm{D}}_{d,t} \quad \forall d, \forall t$$

$$\underline{P}^{\mathrm{G}}_{g,t}\ u^{\mathrm{G,fixed}}_{g,t} \leq p^{\mathrm{G}}_{g,t} \leq \overline{P}^{\mathrm{G}}_{g}\ u^{\mathrm{G,fixed}}_{g,t} \quad \forall g, \forall t$$

$$\sum_{d\in\Psi_n} p^{\mathrm{D}}_{d,t} + \sum_{m\in\Omega_n} B_{n,m}(\theta_{n,t}-\theta_{m,t}) - \sum_{g\in\Psi_n} p^{\mathrm{G}}_{g,t} = 0 \quad : \lambda_{n,t} \quad \forall n, \forall t$$

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$$\theta_{(n=ref),t} = 0 \quad \forall t$$

$$0 \leq s^{\mathrm{G}}_{g,t} \geq C^{\mathrm{SU}}_{g}(u^{\mathrm{G,fixed}}_{g,t} - u^{\mathrm{G,fixed}}_{g,t-1}) \quad \forall g, \forall t$$

By fixing the values of binaries, the optimal values of the start-up cost of generators are somehow given! Their start-up cost variable looks like a parameter in the objective function, and can be eliminated!

Step 2: LP with fixed binaries

$$\underset{p^{\rm G}_{g,t},s^{\rm G}_{g,t},\ p^{\rm D}_{d,t},\ \theta_{n,t}}{\text{Maximize}} \quad \sum_{d,t} U_{d,t}\ p^{\rm D}_{d,t} - \sum_{g,t} [s^{\rm G}_{g,t} + (C_g\ p^{\rm G}_{g,t})]$$

subject to:

$$0 \leq p^{\rm D}_{d,t} \leq \overline{P}^{\rm D}_{d,t} \quad \forall d, \forall t$$

$$\underline{P}^{\rm G}_{g,t}\ u^{\rm G,fixed}_{g,t} \leq p^{\rm G}_{g,t} \leq \overline{P}^{\rm G}_{g}\ u^{\rm G,fixed}_{g,t} \quad \forall g, \forall t$$

$$\sum_{d\in\Psi_n} p^{\rm D}_{d,t} + \sum_{m\in\Omega_n} B_{n,m}(\theta_{n,t} - \theta_{m,t}) - \sum_{g\in\Psi_n} p^{\rm G}_{g,t} = 0 \quad : \lambda_{n,t} \quad \forall n, \forall t$$

$$-F_{n,m} \leq B_{n,m}(\theta_{n,t} - \theta_{m,t}) \leq F_{n,m} \quad \forall n, \forall m \in \Omega_n, \forall t$$

$$\theta_{(n=ref),t} = 0 \quad \forall t$$

$$0 \leq s^{\rm G}_{g,t} \geq C^{\rm SU}_g (u^{\rm G,fixed}_{g,t} - u^{\rm G,fixed}_{g,t-1}) \quad \forall g, \forall t$$

By fixing the values of binaries, the optimal values of the start-up cost of generators are somehow given! Their start-up cost variable looks like a parameter in the objective function, and can be eliminated!

The start-up cost does NOT contribute to the market price formation, because that cost is not reflected in the dual variable!

Step 2: LP with fixed binaries

$$\underset{p^{G}_{g,t},\, s^{G}_{g,t},\, p^{D}_{d,t},\, \theta_{n,t}}{\text{Maximize}} \quad \sum_{d,t} U_{d,t}\, p^{D}_{d,t} - \sum_{g,t} [s^{G}_{g,t} + (C_g\, p^{G}_{g,t})]$$

subject to:

$$0 \leq p^{D}_{d,t} \leq \overline{P}^{D}_{d,t} \quad \forall d, \forall t$$

$$\underline{P}^{G}_{g,t}\, u^{G,\text{fixed}}_{g,t} \leq p^{G}_{g,t} \leq \overline{P}^{G}_{g}\, u^{G,\text{fixed}}_{g,t} \quad \forall g, \forall t$$

$$\sum_{d \in \Psi_n} p^{D}_{d,t} + \sum_{m \in \Omega_n} B_{n,m}(\theta_{n,t} - \theta_{m,t}) - \sum_{g \in \Psi_n} p^{G}_{g,t} = 0 \quad : \lambda_{n,t} \quad \forall n, \forall t$$

$$-F_{n,m} \leq B_{n,m}(\theta_{n,t} - \theta_{m,t}) \leq F_{n,m} \quad \forall n, \forall m \in \Omega_n, \forall t$$

$$) \quad \forall g, \forall t$$

By fixing the values of binaries, the optimal values of the start-up cost of generators are somehow given! Their start-up cost variable looks like a parameter in the objective function, and can be eliminated!

The start-up cost does NOT contribute to the market price formation, because that cost is not reflected in the dual variable!

This means that the "cost recovery" of generators **cannot** be ensured anymore, because market prices do not support their start-up costs!

Step 2: LP with fixed binaries

- In current practice of electricity markets in the U.S., market operators use "**uplift mechanisms**" [1]-[2] to restore "cost recovery", but at the cost of inefficiency and sub-optimality!
- The negative profit of each generator is compensated by an "ex-post" side payment, i.e., "uplift payment".
- The loads will be eventually charged due to uplift payments!

[1] R. O'Neill et al., "Efficient market-clearing prices in markets with nonconvexities," *European Journal of Operational Research*, vol. 164, pp. 269–285, Jul. 2005.

[2] W. W. Hogan and B. J. Ring, "On minimum-uplift pricing for electricity markets," *Harvard Working Paper*, 2003. Available: https://scholar.harvard.edu/whogan/files/minuplift_031903.pdf

Step 2: LP with fixed binaries

- In current practice of electricity markets in the U.S., market operators use "**uplift mechanisms**" [1]-[2] to restore "cost recovery", but at the cost of inefficiency and sub-optimality!
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Discussion:
What is the current practice in the European electricity markets?
Guide: Read about "block orders" and "paradoxically rejected blocks"

Exercise 3

There is another market-clearing mechanism called "Vickrey–Clarke–Groves (VCG)", which pays/charges each generator/demand based on its **value/cost** to the market (see, e.g., references [1]-[2], in the next slide).

Consider a market with three generators G1, G2, G3 and two demands D1 and D2. The market-clearing outcomes in VCG are identical to those in the LMP-based market design; the only difference is "pricing" (but NOT production and consumption levels)!

For example, to calculate the payment to G1 under VCG, the market operator clears the market twice: one time including G1, and another time excluding G1. The payment to G1 is:

$$\text{Payment to G1} = A - B$$

where term A is calculated based on clearing outcomes when G1 exists in the market:
A = [total utility of demands D1 and D2] –[total cost of generators G2 and G3, but **not** G1]

Similarly, term B is calculated based on market-clearing outcomes when G1 is absent:
B = [total utility of demands D1 and D2] –[total cost of generators G2 and G3]

Exercise 3

Question:

- Does VCG mechanism ensure "incentive compatibility", "market efficiency" and "cost recovery"?

Note: The VCG mechanism does <u>not</u> necessarily achieve "revenue adequacy" for the market, which is its main drawback! There are several versions of improved VCG in the literature, seeking to reduce (but not necessarily eliminate) the budget deficit.

[1] Y. Xu and S. H. Low, "An efficient and incentive compatible mechanism for wholesale electricity markets," *IEEE Transactions on Smart Grid*, vol. 8, no. 1, pp. 128-138, 2017.

[2] B. F. Hobbs, M. H. Rothkopf, L. C. Hyde, and R. P. O'Neill, "Evaluation of a truthful revelation auction in the context of energy markets with non-concave benefits," *Journal of Regulatory Economics*, vol. 18, no. 1, pp. 5-32, 2000.

Thanks for your attention!

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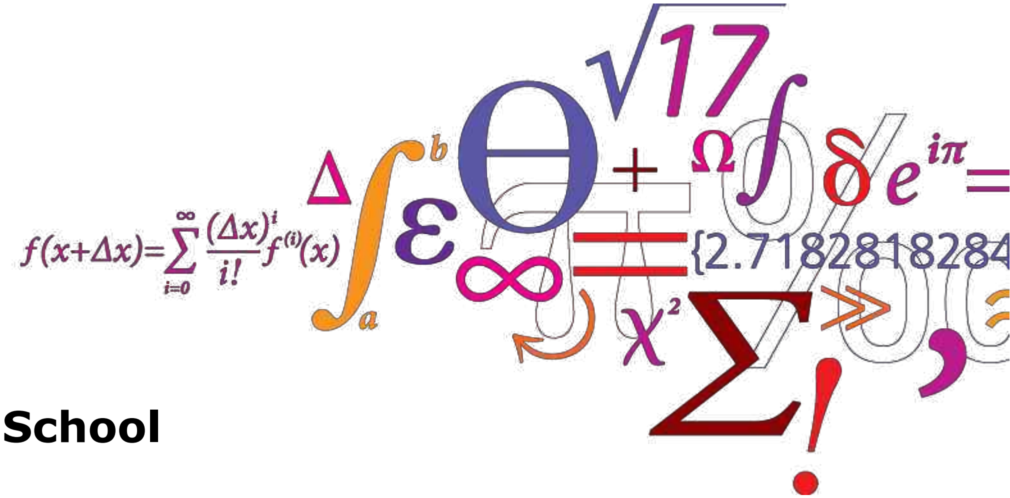
Lecture 4: Bidding strategy in ancillary service markets

Jalal Kazempour

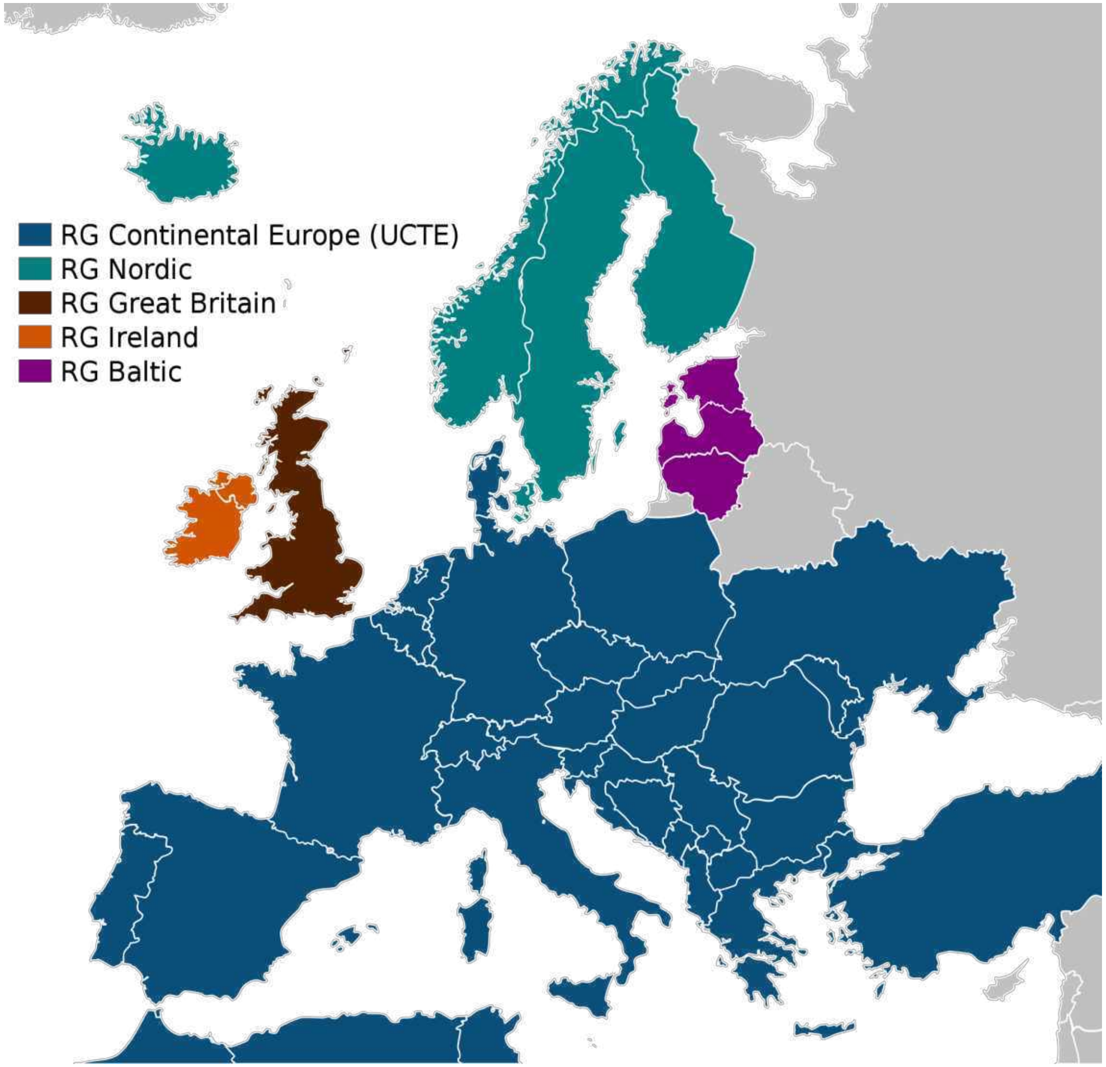


The 6th KIOS Graduate Training School

September 3, 2024

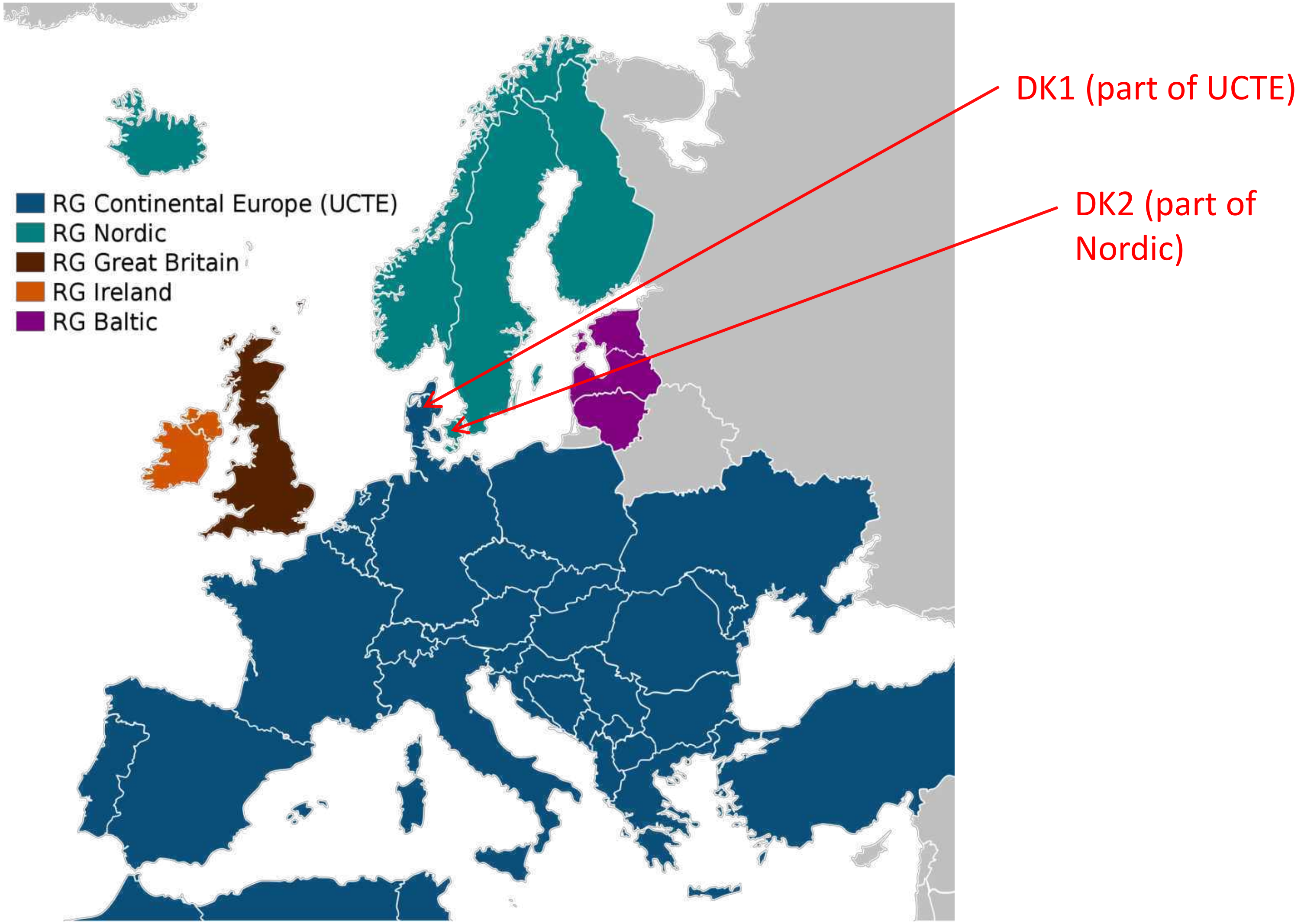
Short introduction to Nordic ancillary service markets

Synchronous grid areas in Europe



Source: *ENTSO-e*

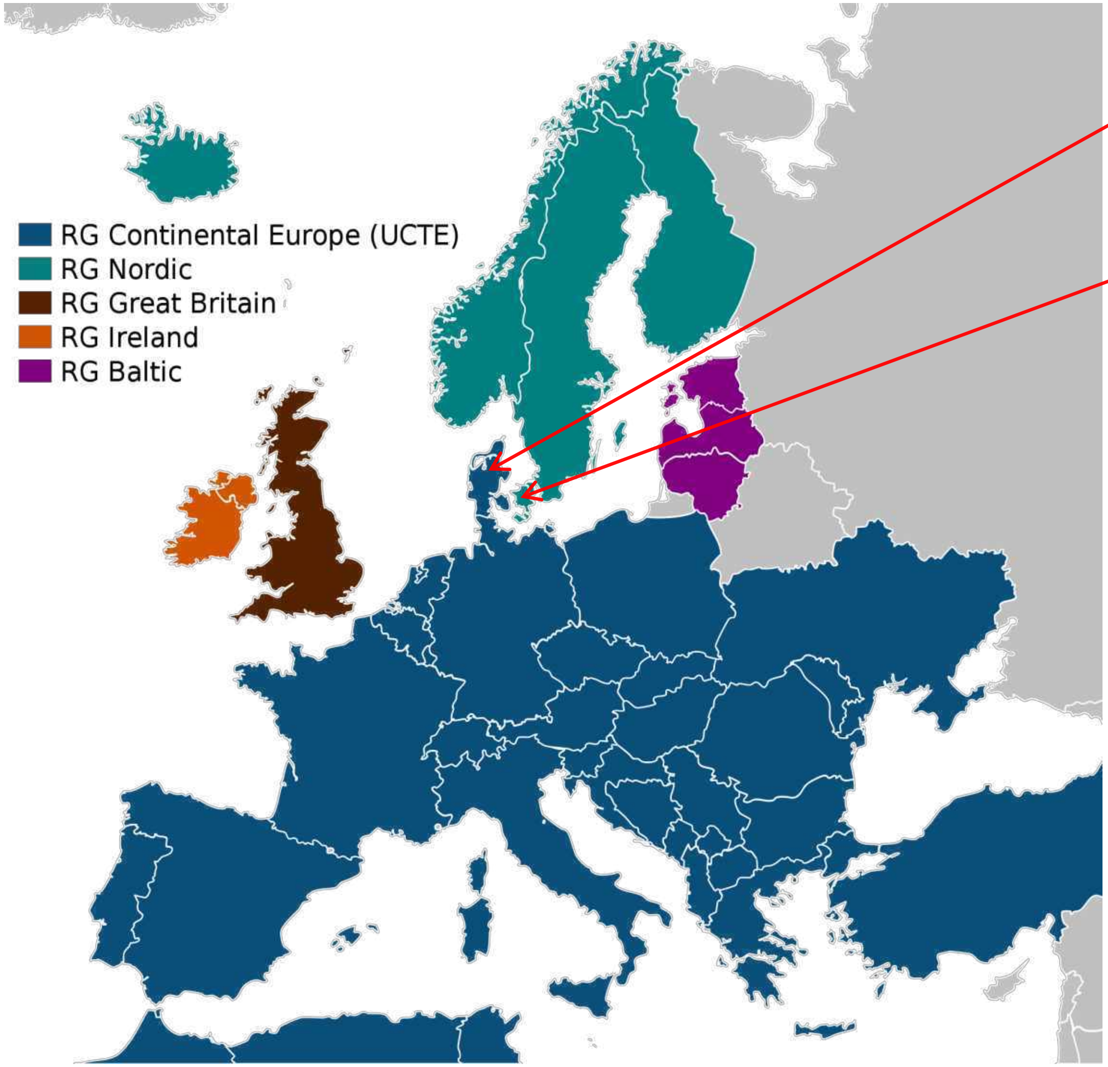
Synchronous grid areas in Europe



Source: *ENTSO-e*

Jalal Kazempour

Synchronous grid areas in Europe



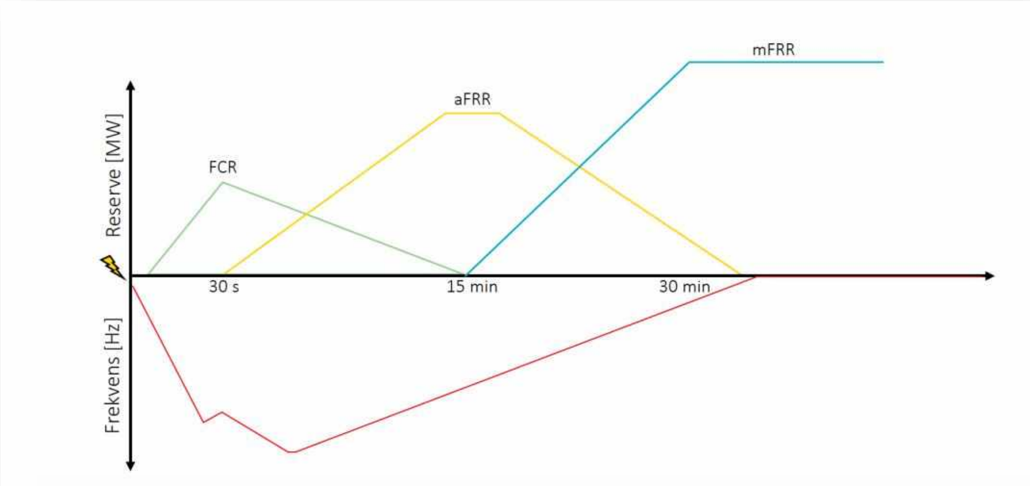
DK1 (part of UCTE)

DK2 (part of Nordic)

Energinet is operating the Danish power system in two areas → different ancillary services exist in DK1 and DK2

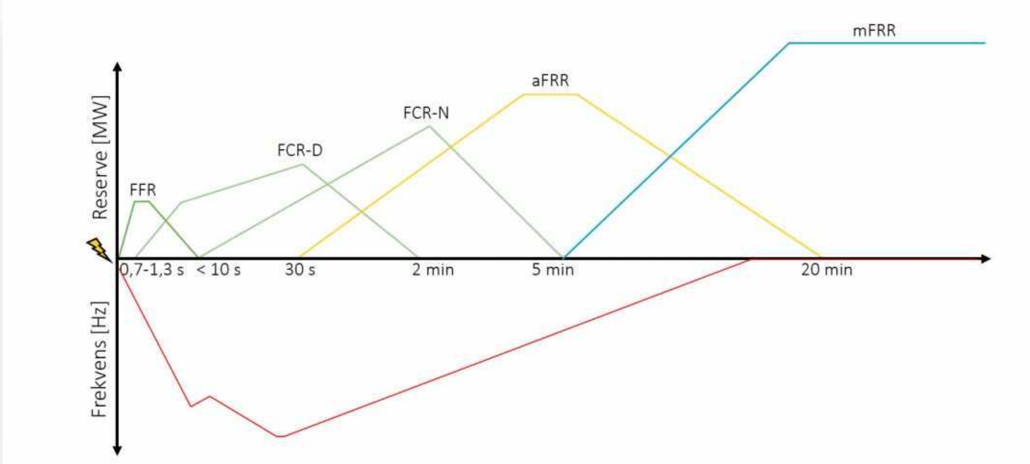
Source: *ENTSO-e*

Frequency-based ancillary services in DK1 and DK2



Ancillary services in DK1 (as part of the continental area)

- FCR
- aFRR
- mFRR



Ancillary services in DK2 (as part of the Nordic area)

- FFR (fast frequency reserve)
- FCR-D (D stands for disturbance)
- FCR-N (N stands for normal)
- aFRR
- mFRR

Source: *Energinet (Gennemgang af Nuværende Systemydelse Markeder)*

Specifics of ancillary services in DK1 and DK2

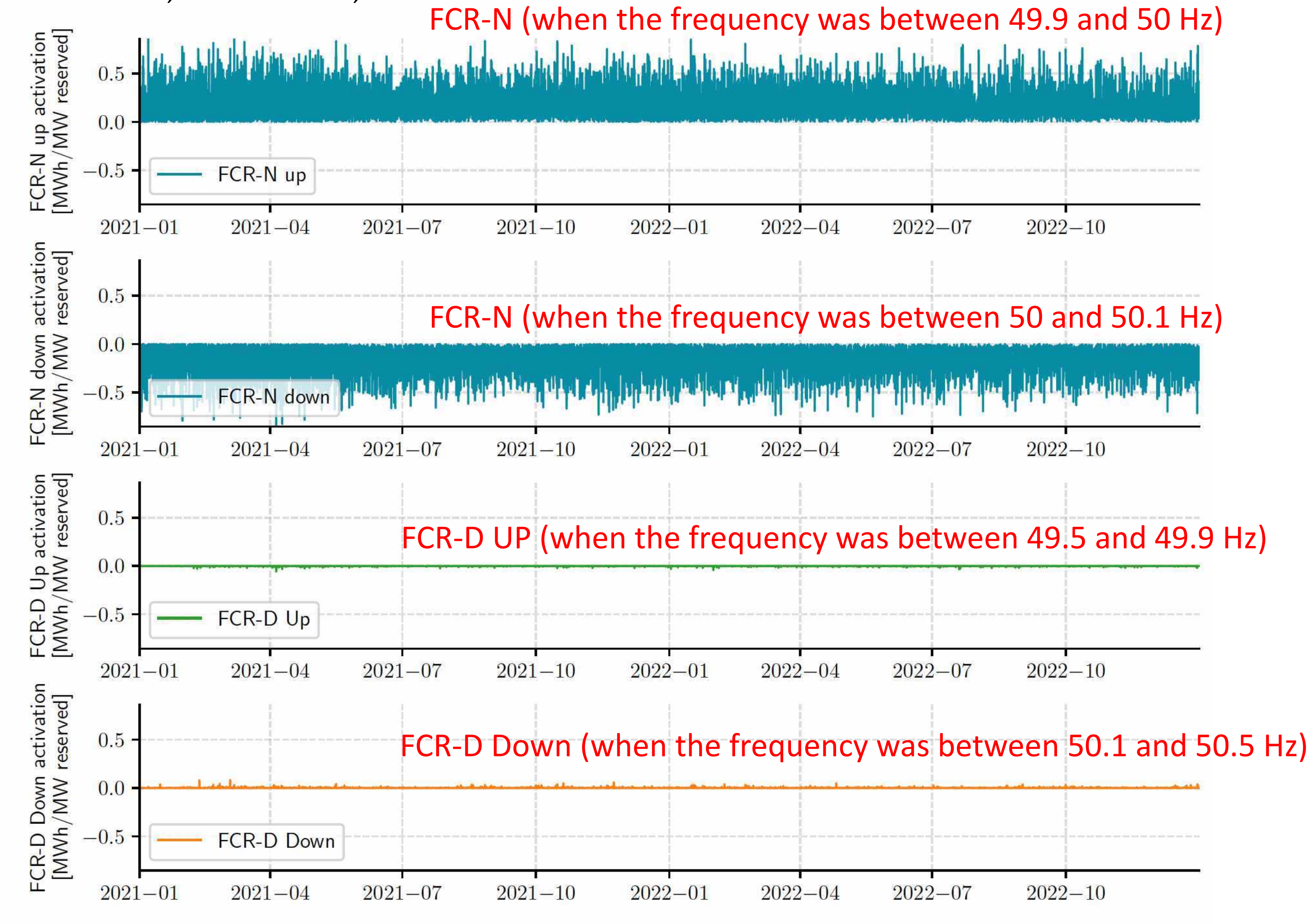
Source: *Energinet (Gennemgang af Nuværende Systemydelse Markeder)*

Potential service providers in DK1 and DK2

Source: *Energinet (Gennemgang af Nuværende Systemydelse Markeder)*

Historical data: Activated FCR-D and FCR-N in DK2 (2021-2022)

Credit: *Marco Saretta, DTU MSc thesis, 2023*



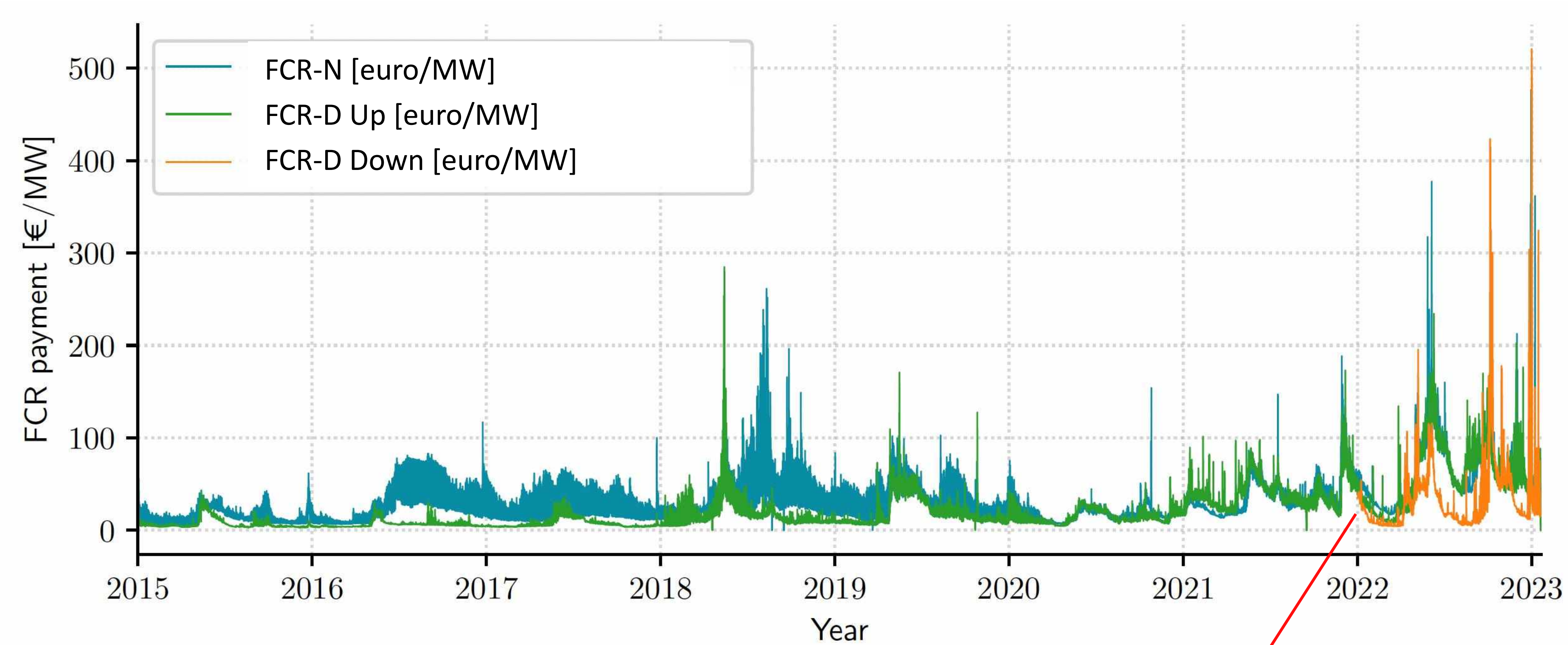
Historical data: Activated FCR-D and FCR-N in DK2 (2021-2022)

Credit: *Marco Saretta, DTU MSc thesis, 2023*

FCR-D was very rarely activated! Service providers received payments due to capacity reservation but were activated very rarely!

Historical data: FCR-D and FCR-N prices in DK2 (2015-2022)

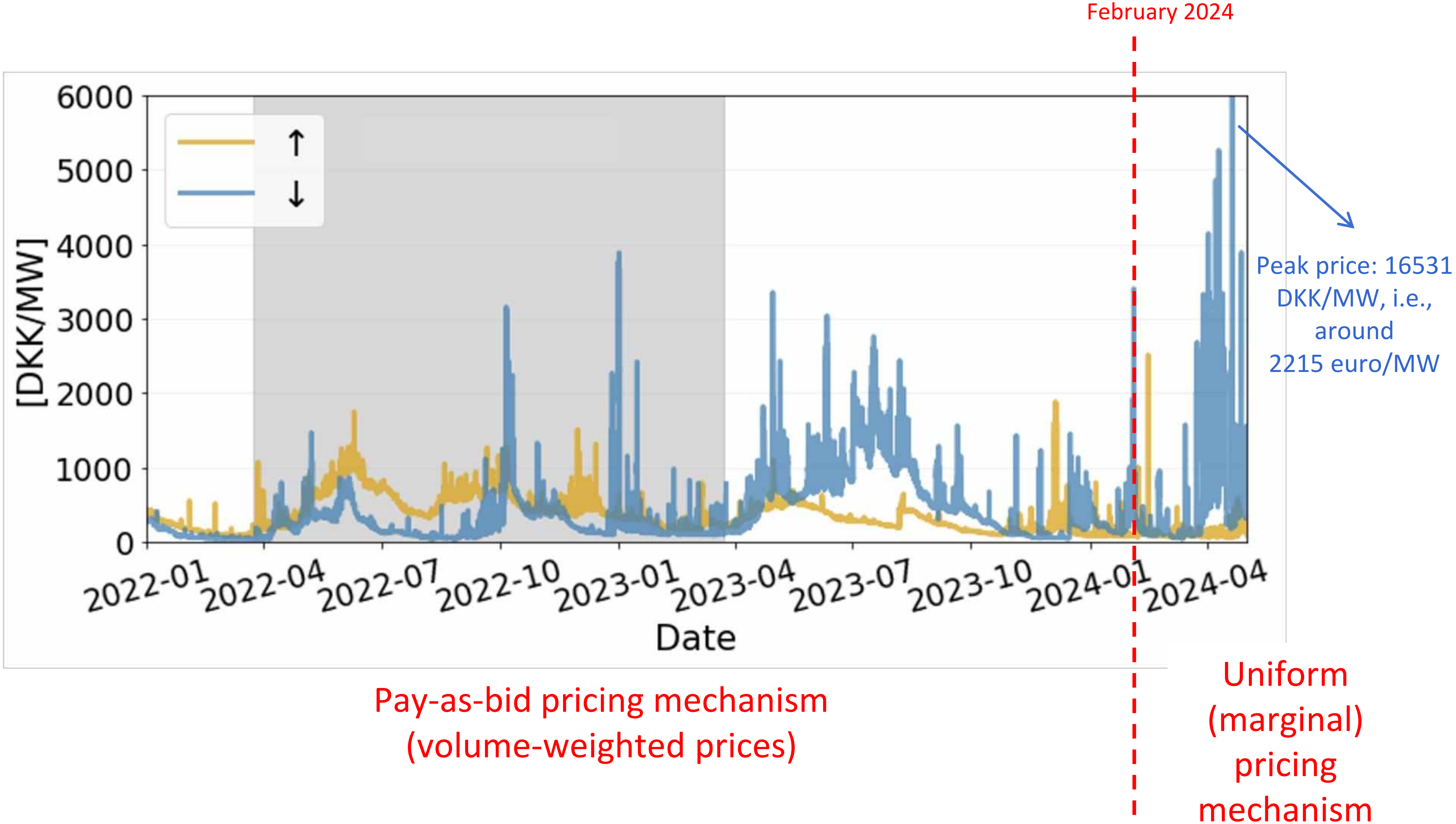
Credit: *Marco Saretta, DTU MSc thesis, 2023*



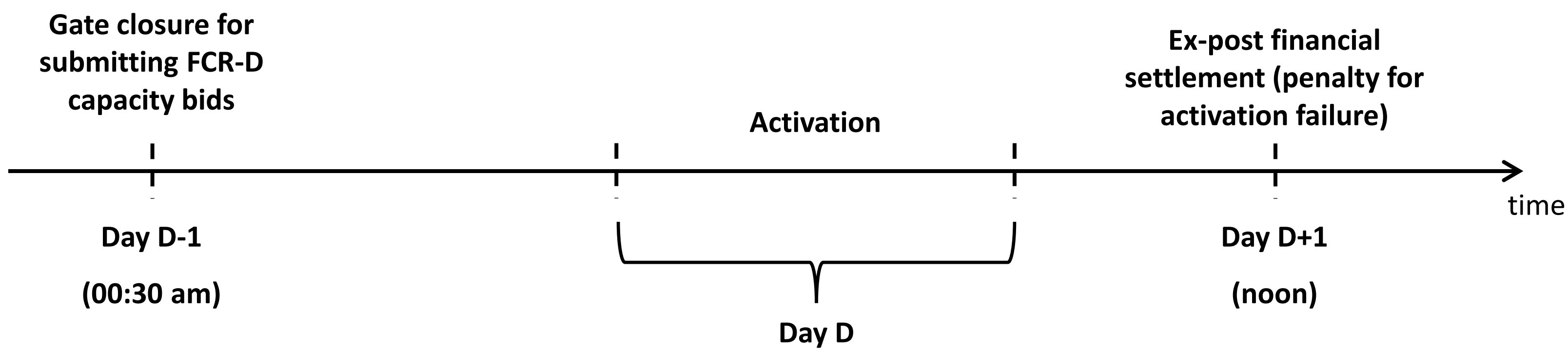
FCR-D Down started in January 2022

A closer look at historical FCR-D up/down prices in Denmark (DK2)

Credit: *Gustav Lunde and Emil Damm, DTU MSc thesis, 2024*



Current market for FCR-D Up/Down in Denmark



- The FCR-D services are used to be bought in D-2 (until very recently). Now it is in D-1.
- There is a second (optional) market for FCR-D in D-1 in case TSOs realize more FCR-D services should be bought.
- Payment for capacity only (activation is not “energy-intensive”)
- Penalty for activation failure = the cost of alternative source

Nordic TSO obligations to procure FCR services in 2023

	Share [%]	FCR-N [MW]	FCR-D Up [MW]	FCR-D Down [MW]
StatNett	39	234	564	546
FinGrid	20	120	290	280
Svenska Kraftnat	38.3	230	555	536
Energinet	2.7	17	41	38
Nordic obligations	100	600	1450	1400

Source: Energinet report [link]
Outlook for the need in 2030-2040: Energinet report [link]
Credit: *Marco Saretta, DTU MSc thesis, 2023*
Relevant article by Marco et al: [link]

Stochastic flexible assets that can bid their flexibility to ancillary service markets

- These assets could be in the demand or supply side!
- Examples of stochastic flexible assets: electric vehicles (EVs), heat pumps, supermarket freezers, wind turbines, etc.
- Future consumption/production level of these assets is stochastic → stochastic baseline!
- Without loss of generalization, from now on, we consider the FCR-D Up/Down market as our example ancillary service markets!

Data for an aggregation of EV charging boxes in Denmark

Credit: *Gustav Lunde and Emil Damm, DTU MSc thesis, 2024*

- Data for electric vehicles (EVs) provided by Spirii (https://spirii.com/en)
- Time period of March 24, 2022, to March 21, 2023
- Minute-level resolution (the ideal is to have a higher-resolution dataset)

Minute

Data for an aggregation of EV charging boxes in Denmark

Credit: *Gustav Lunde and Emil Damm, DTU MSc thesis, 2024*

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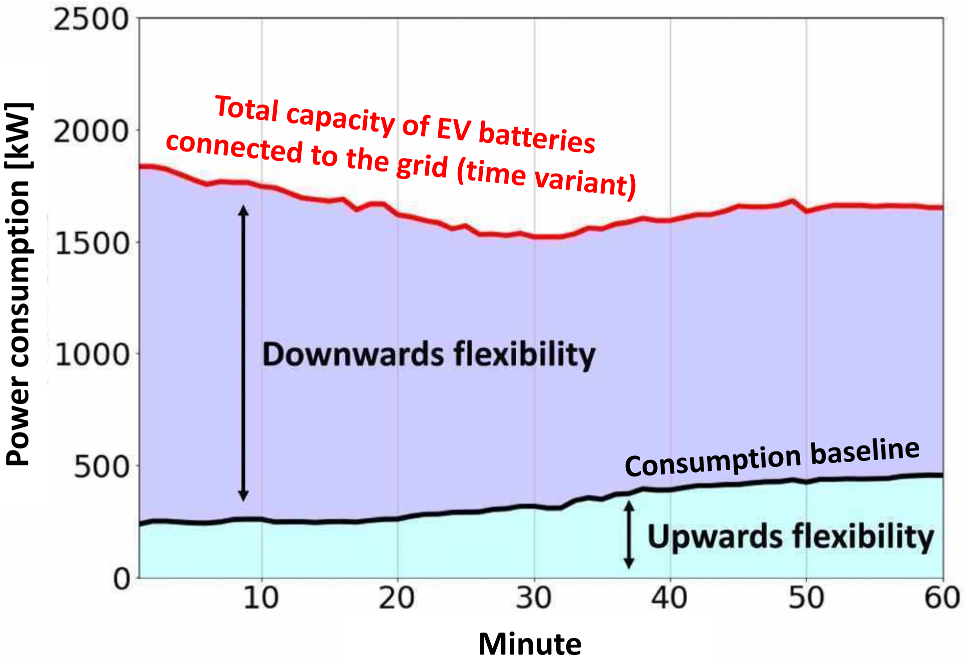
Consumption level for a random historical hour (an aggregation of 1400 EV charging boxes)



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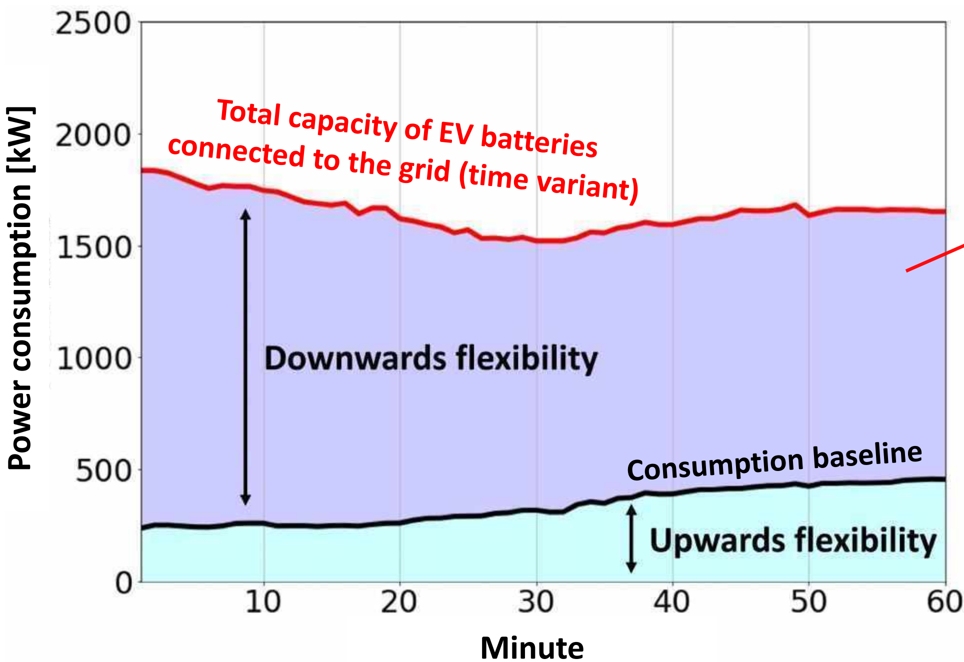

Potential for down-regulation (e.g., FCR-D Down services) by increasing the consumption level

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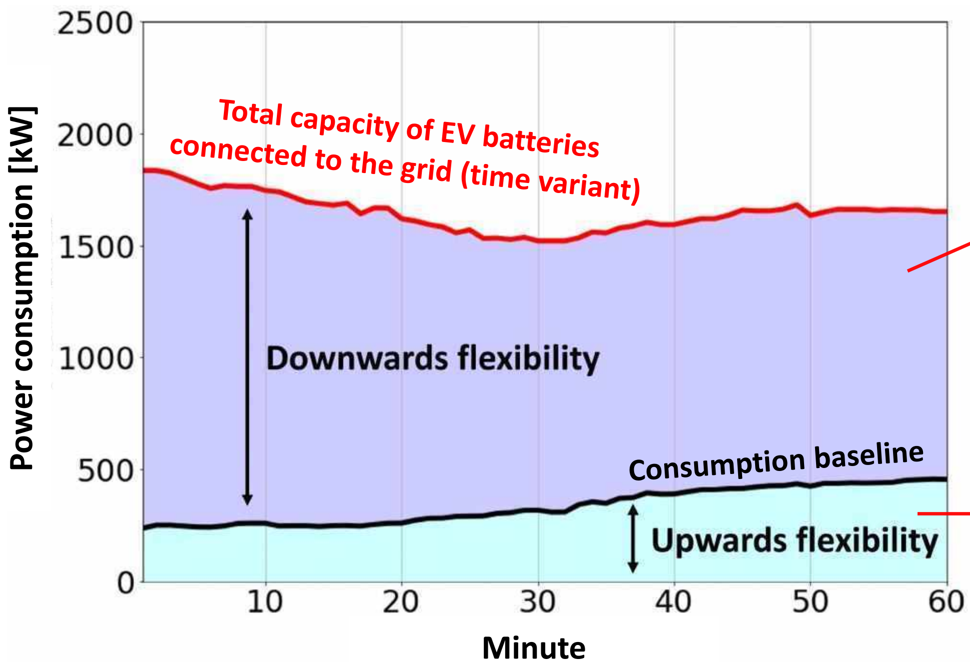


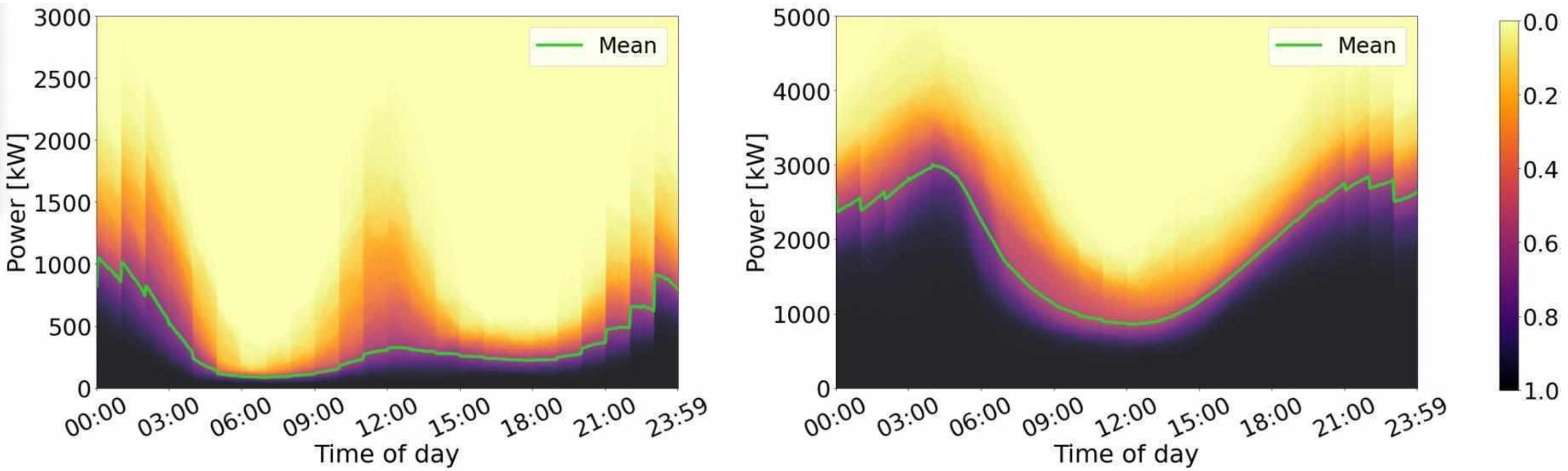
Potential for down-regulation (e.g., FCR-D Down services) by increasing the consumption level

Potential for up-regulation (e.g., FCR-D UP services) by decreasing the consumption level without any Vehicle-to-Grid (V2G) option --> no power being injected from the battery to the grid

Data for an aggregation of EV charging boxes in Denmark

Credit: *Gustav Lunde and Emil Damm, DTU MSc thesis, 2024*

1-CDF of potential [kW] for FCR-D Up (left plot) and FCR-D Down (right plot) services throughout the day (based on historical data for 1400 EV charging boxes). CDF = cumulative distribution function.



(a) Upwards flexibility $F^{\uparrow}$ (b) Downwards flexibility $F^{\downarrow}$

Data for an aggregation of EV charging boxes in Denmark

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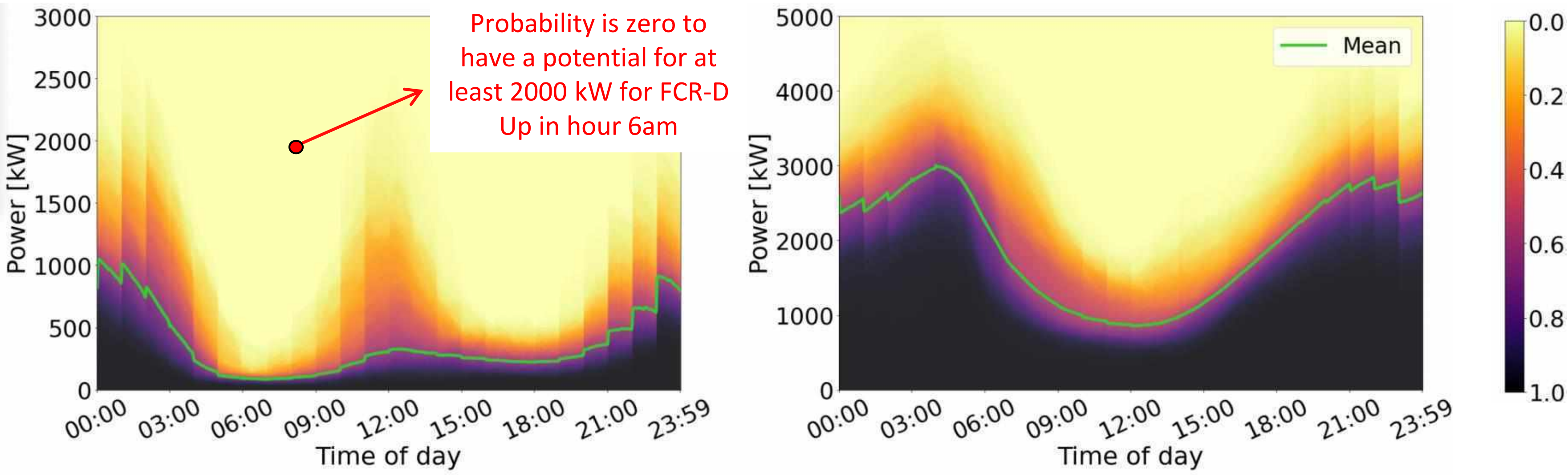

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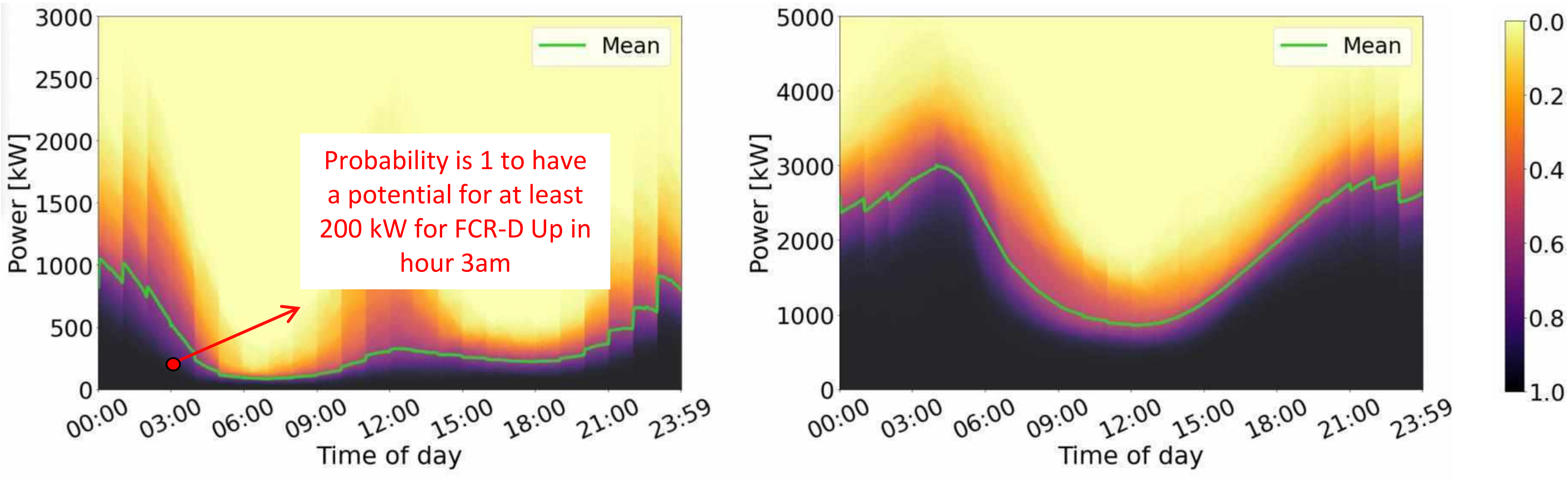

(a) Upwards flexibility $F^{\uparrow}$

(b) Downwards flexibility $F^{\downarrow}$

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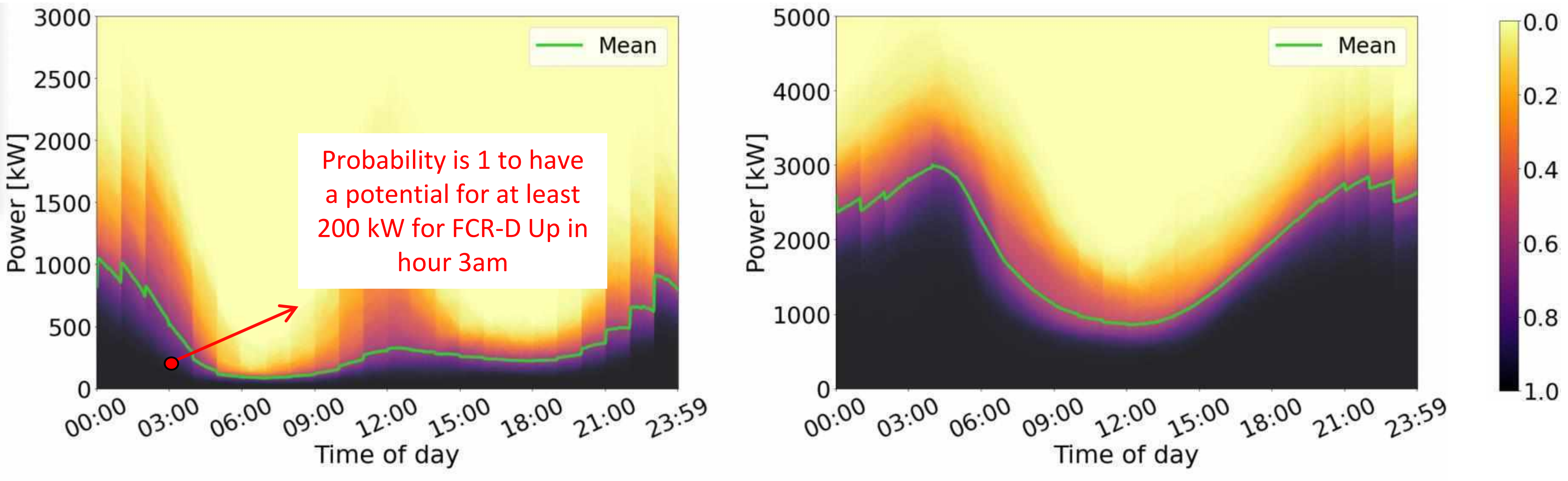

Note:
These two distribution functions are built based on available historical data during the period of March 24, 2022, to March 21, 2023. It is not necessarily the best way to utilize data. For example, one may use these data to "probabilistically forecast" the future baseline and then use it for bidding decision-making purposes. Or due to seasonality or non-stationarity reasons or alike, one may use the most recent/relevant data for the representation of baseline for the next day!

The P90 requirement of Energinet

- The name “P90” was given by us. It is not used in the Energinet report.

- “Energinet: Prequalification and test,” Energinet, 2023, accessed: 2024-05-30. [Online]. Available: https://en.energinet.dk/electricity/ancillary-services/prequalification-and-test/

The P90 requirement

"Energinet requires that there must at maximum be bid in capacity corresponding to the 10% percentile with delivery of capacity reserves from fluctuating renewables and flexible consumption. This means, that the participant's prognosis, which must be approved by Energinet, evaluates that the probability is 10% that the sold capacity is not available. This entails that there is a 90% chance that the sold capacity or more is available. This is when the prognosis is assumed to be correct. The probability is then also 10\%, that the entire sold capacity is not available. If this were to happen, it does not entail that the sold capacity is not available at all, however just that a part of the total capacity is not available. The available part will with a high probability be close to the sold capacity."

Source: Energinet report [link]

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Source: Energinet report [link]

How do you interpret this requirement?

The P90 requirement

"Energinet requires that there must at maximum be bid in capacity corresponding to the 10% percentile with delivery of capacity reserves from fluctuating renewables and flexible consumption. This means, that the participant's prognosis, which must be approved by Energinet, evaluates that the probability is 10% that the sold capacity is not available. This entails that there is a 90% chance that the sold capacity or more is available. This is when the prognosis is assumed to be correct. The probability is then also 10\%, that the entire sold capacity is not available. If this were to happen, it does not entail that the sold capacity is not available at all, however just that a part of the total capacity is not available. The available part will with a high probability be close to the sold capacity."

Source: Energinet report [link]

This requirement lets a stochastic recourse bid in Nordic ancillary service markets, provided the probability of the bid to be successfully realized is at least 90% -- this means the resource will be still counted qualified for bidding in ancillary service markets if the probability of "**reserve shortfall"** (also called "**overbidding"**) is not more than 10%.

The P90 requirement

"Energinet requires that there must at maximum be bid in capacity corresponding to the 10% percentile with delivery of capacity reserves from fluctuating renewables and flexible consumption. This means, that the participant's prognosis, which must be approved by Energinet, evaluates that the probability is 10% that the sold capacity is not available. This entails that there is a 90% chance that the sold capacity or more is available. This is when the prognosis is assumed to be correct. The probability is then also 10\%, that the entire sold capacity is not available. If this were to happen, it does not entail that the sold capacity is not available at all, however just that a part of the total capacity is not available. The available part will with a high probability be close to the sold capacity."

Source: Energinet report [link]

How does Energinet check this requirement for given bids?

The P90 requirement

How does Energinet check it?

Harry van der Weijde (He/Him) • 1st · 1d
Senior Scientist at TNO Vector | Energy | Transport | Economics | S...

Interesting! How is the requirement enforced, since only the realisations are visible and the underlying distribution is usually not? How do bidders prove a 90% probability?

Like · 1 | Reply · 1 Reply

Thomas Dalgas Fechtenburg • 1st · 1d
Senior Manager, Ancillary Services, Energinet

Harry van der Weijde - based on at least three months of historical performance, where the P90 proved to be available at least 15% of the time (a binary consideration). We continuously monitor the performance of both the physical delivery and forecasts as well, which allow for a "low" entry criteria.

Like · 2 | Reply

Thomas Dalgas Fechtenburg • 1st · 2d
Senior Manager, Ancillary Services, Energinet

I'm glad you find our requirement interesting! After having it for ~3 years now, we start to see the effect of it. From our perspective it took some time to learn, but now multiple providers have developed probabilistic forecasts to meet it effectively.
Looking forward to read your paper!

Like · 1 | Reply · 1 Reply

Jalal Kazempour Author · 2d
Head of Section, Head of Studies, Associate Professor at ...

Thanks Thomas for the comment and all discussions we had so far. Indeed it is very innovative and interesting. I am not aware of any other TSO with a similar innovative requirement. Very nice to hear there are now some flex providers meeting

Mathematical representation of the P90 requirement

Let's start with the FCR-D Up market. One can formulate a similar optimization problem for the FCR-D Down market.

Bidding a reserve capacity (in kW) to the FCR-D Up market in a given hour (say hour *h*):

$$\underset{c^{\uparrow}\geq 0}{\text{Max}} \quad c^{\uparrow}$$

subject to:

$$\mathbb{P}\left(c^{\uparrow} \leq F_m^{\uparrow}, \quad \forall m\right) \geq 1-\epsilon$$

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$$\underset{c^{\uparrow} \geq 0}{\text{Max}} \quad c^{\uparrow}$$

Reserve capacity bid [kW] in the given hour to be offered to the FCR-D Up market. This is our decision variable!

subject to:

$$\mathbb{P}\left(c^{\uparrow} \leq F_m^{\uparrow}, \quad \forall m\right) \geq 1 - \epsilon$$

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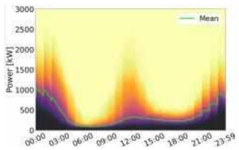
Probability distribution of the FCR-D Up service availability per minute *m* of the given hour

Minutes = {1, 2, …, 60} in the given hour *h*



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0.1 as per the P90 requirement

P(.): Probability function.
Note that we have a "**probabilistic constraint**"!

Mathematical representation of the P90 requirement

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What does this "**probabilistic constraint**" enforce?

Mathematical representation of the P90 requirement

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$$\underset{c^{\uparrow} \geq 0}{\text{Max}} \quad c^{\uparrow}$$

subject to:

$$\mathbb{P}\left(c^{\uparrow} \leq F^{\uparrow}_{m}, \quad \forall m\right) \geq 1 - \epsilon$$

What does this "**probabilistic constraint**" enforce?

It enforces the "probability" of the set of constraints inside P(.) to be met should be at least 90%. Given 60 minutes, it enforces our reserve capacity bid corresponding to hour *h* should be available at least in 54 minutes of that hour and we should not see a reserve shortfall in more than 6 minutes! It does not say about the "magnitude" of shortfall though as it is the case in the P90 requirement too.

Mathematical representation of the P90 requirement

Let's start with the FCR-D Up market. One can formulate a similar optimization problem for the FCR-D Down market.

Bidding a reserve capacity (in kW) to the FCR-D Up market in a given hour (say hour *h*):

$$\underset{c^{\uparrow} \geq 0}{\text{Max}} \quad c^{\uparrow}$$

subject to:

$$\mathbb{P}\left(c^{\uparrow} \leq F_m^{\uparrow}, \quad \forall m\right) \geq 1 - \epsilon$$

What does this "**probabilistic constraint**" enforce?

- This is a "**chance-constrained program**"! It is a well-known class of optimization problems under uncertainty!
- This is specifically a "**joint**" chance-constrained program as we have more than one constraint within P(.).

Mathematical representation of the P90 requirement

Let's start with the FCR-D Up market. One can formulate a similar optimization problem for the FCR-D Down market.

Bidding a reserve capacity (in kW) to the FCR-D Up market in a given hour (say hour *h*):

$$\underset{c^{\uparrow}\geq 0}{\text{Max}} \quad c^{\uparrow}$$

subject to:

$$\mathbb{P}\left(c^{\uparrow} \leq F_m^{\uparrow}, \quad \forall m\right) \geq 1-\epsilon$$

What does this "**probabilistic constraint**" enforce?

Question: Having data with a second-resolution than minutes, does it make our decision-making optimization problem more flexible (and less conservative)?

How to solve a (joint) chance-constrained program?

Two solution techniques:

1. **ALSO-X** (reference [1]-[2]. ALSO-X is the initials of co-authors in [1].)
2. **Conditional value-at-risk (CVaR) approximation**

Both techniques require ***sampling*** from distributions. Recall we have 60 distributions, one per minute. We draw $w = \{w1, w2, \ldots, |w|\}$ arbitrary samples from each distribution.

[1] S. Ahmed, J. Luedtke, Y. Song, and W. Xie, "Nonanticipative duality, relaxations, and formulations for chance-constrained stochastic programs," *Mathematical Programming*, vol. 162, no. 1, pp. 51–81, 2017.

[2] N. Jiang and W. Xie, "ALSO-X and ALSO-X+: Better convex approximations for chance constrained programs" *Operations Research*, vol. 70, no. 6, pp. 3581–3600, 2022.

How to solve a (joint) chance-constrained program?

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If the underlying probability distribution admits certain properties, we can have an "**analytical**" reformulation [3] → Satisfactory out-of-sample performance.

[3] A. Nemirovski and A. Shapiro, "Convex approximations of chance constrained programs," *SIAM Journal on Optimization*, vol. 17, no. 4, pp. 969–996, 2007.

How to solve a (joint) chance-constrained program?

Two solution techniques:

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Both techniques require ***sampling*** from distributions. Recall we have 60 distributions, one per minute. We draw $w = \{w1, w2, \ldots, |w|\}$ arbitrary samples from each distribution.

➢ What is the minimum number of samples that we should use? See [4] for the answer!

[4] J. Luedtke and S. Ahmed, "A sample approximation approach for optimization with probabilistic constraints," *SIAM Journal of Optimization*, vol. 19, no. 2, pp. 674-699, 2008.

ALSO-X

Chance constraint → sample-based MILP reformulation → LP relaxation → iterative algorithm

ALSO-X solution technique

$$\max_{c^{\uparrow} \geq 0} \quad c^{\uparrow}$$

subject to:

$$\mathbb{P}\left(c^{\uparrow} \leq F_m^{\uparrow}, \quad \forall m\right) \geq 1 - \epsilon$$

Joint chance-constrained program

ALSO-X solution technique

$$\underset{c^{\uparrow}\geq 0}{\text{Max}} \quad c^{\uparrow}$$

subject to:

$$\mathbb{P}\left(c^{\uparrow} \leq F_m^{\uparrow}, \quad \forall m\right) \geq 1-\epsilon$$

Joint chance-constrained program

Reformulation based on samples

ALSO-X solution technique

$$\underset{c^{\uparrow}\geq 0}{\text{Max}} \quad c^{\uparrow}$$

subject to:

$$\mathbb{P}\left(c^{\uparrow} \leq F_m^{\uparrow}, \quad \forall m\right) \geq 1-\epsilon$$

Joint chance-constrained program

Reformulation based on samples

$$\underset{c^{\uparrow}\geq 0, y_{m,\omega}\in\{0,1\}}{\text{Max}} \qquad c^{\uparrow}$$

subject to:

$$c^{\uparrow} - F_{m,\omega}^{\uparrow} \leq y_{m,\omega} M \quad \forall m, \omega$$

$$\sum_m \sum_\omega y_{m,\omega} \leq q$$

Sample-based mixed-integer linear program (MILP)

ALSO-X solution technique

$$\underset{c^{\uparrow}\geq 0}{\text{Max}} \quad c^{\uparrow}$$

subject to:

$$\mathbb{P}\left(c^{\uparrow} \leq F_m^{\uparrow}, \quad \forall m\right) \geq 1-\epsilon$$

Unchanged.
Not indexed by sample w.

$$\underset{c^{\uparrow}\geq 0, y_{m,\omega}\in\{0,1\}}{\text{Max}} \quad c^{\uparrow}$$

subject to:

$$c^{\uparrow} - F_{m,\omega}^{\uparrow} \leq y_{m,\omega} M \quad \forall m, \omega$$

$$\sum_{m}\sum_{\omega} y_{m,\omega} \leq q$$

ALSO-X solution technique

$$\underset{c^{\uparrow}\geq 0}{\text{Max}} \quad c^{\uparrow}$$

subject to:

$$\mathbb{P}\left(c^{\uparrow} \leq F^{\uparrow}_{m}, \quad \forall m\right) \geq 1-\epsilon$$

$$\underset{c^{\uparrow}\geq 0, y_{m,\omega}\in\{0,1\}}{\text{Max}} \quad c^{\uparrow}$$

Binary variables, one per minute, per sample!

subject to:

$$c^{\uparrow} - F^{\uparrow}_{m,\omega} \leq y_{m,\omega} M \quad \forall m, \omega$$

$$\sum_{m}\sum_{\omega} y_{m,\omega} \leq q$$

ALSO-X solution technique

$$\underset{c^{\uparrow} \geq 0}{\text{Max}} \quad c^{\uparrow}$$

subject to:

$$\mathbb{P}\left(c^{\uparrow} \leq F_m^{\uparrow}, \quad \forall m\right) \geq 1 - \epsilon$$

$$\underset{c^{\uparrow} \geq 0, y_{m,\omega} \in \{0,1\}}{\text{Max}} \quad c^{\uparrow}$$

subject to:

Sample *w* from the distribution for minute *m*

$$c^{\uparrow} - F_{m,\omega}^{\uparrow} \leq y_{m,\omega} M \quad \forall m, \omega$$

$$\sum_{m} \sum_{\omega} y_{m,\omega} \leq q$$

ALSO-X solution technique

$$\underset{c^{\uparrow} \geq 0}{\text{Max}} \quad c^{\uparrow}$$

subject to:

$$\mathbb{P}\left(c^{\uparrow} \leq F_m^{\uparrow}, \quad \forall m\right) \geq 1 - \epsilon$$

$$\underset{c^{\uparrow} \geq 0, y_{m,\omega} \in \{0,1\}}{\text{Max}} \quad c^{\uparrow}$$

subject to:

$$c^{\uparrow} - F_{m,\omega}^{\uparrow} \leq y_{m,\omega} M \quad \forall m, \omega$$

$$\sum_m \sum_\omega y_{m,\omega} \leq q$$

A large enough positive constant, e.g., 10000.

ALSO-X solution technique

$$\underset{c^{\uparrow} \geq 0}{\text{Max}} \quad c^{\uparrow}$$

subject to:

$$\mathbb{P}\left(c^{\uparrow} \leq F^{\uparrow}_{m}, \quad \forall m\right) \geq 1 - \epsilon$$

$$\underset{c^{\uparrow} \geq 0, y_{m,\omega} \in \{0,1\}}{\text{Max}} \quad c^{\uparrow}$$

subject to:

$$c^{\uparrow} - F^{\uparrow}_{m,\omega} \leq y_{m,\omega} M \quad \forall m, \omega$$

$$\sum_{m} \sum_{\omega} y_{m,\omega} \leq q$$

Indicating whether the probabilistic constraint in the original problem has been "**violated**" in minute *m* under sample *w:*
y=0 → no
y=1 → yes

ALSO-X solution technique

$$\underset{c^{\uparrow}\geq 0}{\text{Max}} \quad c^{\uparrow}$$

subject to:

$$\mathbb{P}\left(c^{\uparrow} \leq F_{m}^{\uparrow}, \quad \forall m\right) \geq 1-\epsilon$$

$$\underset{c^{\uparrow}\geq 0, y_{m,\omega}\in\{0,1\}}{\text{Max}} \quad c^{\uparrow}$$

subject to:

$$c^{\uparrow} - F_{m,\omega}^{\uparrow} \leq y_{m,\omega} M \quad \forall m, \omega$$

$$\sum_{m} \sum_{\omega} y_{m,\omega} \leq q$$

Total cases (for all minutes and samples) the original probabilistic constraint has been "**violated**"!

ALSO-X solution technique

$$\underset{c^{\uparrow}\geq 0}{\text{Max}} \quad c^{\uparrow}$$

subject to:

$$\mathbb{P}\left(c^{\uparrow} \leq F_m^{\uparrow}, \quad \forall m\right) \geq 1-\epsilon$$

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subject to:

$$c^{\uparrow} - F_{m,\omega}^{\uparrow} \leq y_{m,\omega} M \quad \forall m, \omega$$

$$\sum_{m}\sum_{\omega} y_{m,\omega} \leq q$$

Our budget for violation (given parameter)
= 10% * number of samples * number of minutes

ALSO-X solution technique

The challenge of this problem is the number of binary variables. With 1000 samples, we will have 60,000 binary variables, which may make the problem **computationally expensive or even intractable!**

$$\underset{c^{\uparrow}\geq 0,\, y_{m,\omega}\in\{0,1\}}{\text{Max}} \quad c^{\uparrow}$$

subject to:

$$c^{\uparrow} - F^{\uparrow}_{m,\omega} \leq y_{m,\omega} M \quad \forall m,\omega$$

$$\sum_{m}\sum_{\omega} y_{m,\omega} \leq q$$

MILP

ALSO-X solution technique

Algorithm 1 ALSO-X

Input: Stopping tolerance parameter δ, e.g., $\delta = 10^{-5}$

Require: Relax the integrality of y

1: $\underline{q} \leftarrow 0$

$\bar{q} \leftarrow \epsilon \times$ number of samples $\times$ number of samples

2: **while** $\bar{q} - \underline{q} \geq \delta$ **do**

3: Set $q = \frac{(\underline{q}+\bar{q})}{2}$

4: Retrieve Θ^* as an optimal solution to the relaxed problem, i.e., the LP.

5: Set $\underline{q} = q$ if $\mathbb{P}(y_{m,\omega}* = 0) \geq 1 - \epsilon$; otherwise, $\bar{q} = q$

6: **end while**

Output: A feasible solution to the non-relaxed problem, i.e., the MILP.

Let's relax every binary variable between zero and one (so, MILP→ LP) and solve an iterative algorithm the so-called ALSO-X algorithm!

$$\max_{c^{\uparrow} \geq 0, y_{m,\omega} \in \{0,1\}} \quad c^{\uparrow}$$

subject to:

$$c^{\uparrow} - F^{\uparrow}_{m,\omega} \leq y_{m,\omega} M \quad \forall m, \omega$$

$$\sum_{m} \sum_{\omega} y_{m,\omega} \leq q$$

MILP

CVaR

Chance constraint → CVaR constraint (conservative approximation of chance constraint) → sample-based convex reformulation

CVaR reformulation

- The CVaR method [4] approximates the joint chance constraint by controlling *magnitude* of reserve shortfall using a reformulated LP. This is why the CVaR reformulation is <u>more conservative</u> than the original chance-constrained problem.

[4] R. T. Rockafellar and S. Uryasev, "Optimization of conditional value-at-risk," *Journal of Risk*, vol. 2, pp. 21–42, 2000

CVaR reformulation

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- Thus, the CVaR minimizes the expected reserve shortfall for the worst (1-\epsilon) samples which is the value-at-risk (VaR). Recall \epsilon = 0.1 as per the P90 requirement.

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- Thus, the CVaR minimizes the expected reserve shortfall for the worst (1-\epsilon) samples which is the value-at-risk (VaR). Recall \epsilon = 0.1 as per the P90 requirement.
- The CVaR approximation problem reads as

$$\underset{c^{\uparrow}\geq 0, \beta \leq 0, \zeta_{m,\omega}}{\text{Max}} \quad c^{\uparrow}$$

$$\text{subject to:}$$

$$c^{\uparrow} - F^{\uparrow}_{m,\omega} \leq \zeta_{m,\omega} \quad \forall m, \omega$$

$$\frac{1}{|m||\omega|}\sum_{m}\sum_{\omega}\zeta_{m,\omega} \leq (1-\epsilon)\beta$$

$$\beta \leq \zeta_{m,\omega} \quad \forall m, \omega$$

[4] R. T. Rockafellar and S. Uryasev, “Optimization of conditional value-at-risk,” *Journal of Risk*, vol. 2, pp. 21–42, 2000

CVaR reformulation

- The CVaR method [4] approximates the joint chance constraint by controlling *magnitude* of reserve shortfall using a reformulated LP. This is why the CVaR reformulation is more conservative than the original chance-constrained problem.
- Thus, the CVaR minimizes the expected reserve shortfall for the worst (1-\epsilon) samples which is the value-at-risk (VaR). Recall \epsilon = 0.1 as per the P90 requirement.
- The CVaR approximation problem reads as

$$\underset{c^{\uparrow}\geq 0, \beta \leq 0, \zeta_{m,\omega}}{\text{Max}} \quad c^{\uparrow}$$

subject to:

$$c^{\uparrow} - F^{\uparrow}_{m,\omega} \leq \zeta_{m,\omega} \quad \forall m, \omega$$

$$\frac{1}{|m||\omega|} \sum_{m} \sum_{\omega} \zeta_{m,\omega} \leq (1-\epsilon)\beta$$

number of samples * number of minutes

$$\beta \leq \zeta_{m,\omega} \quad \forall m, \omega$$

[4] R. T. Rockafellar and S. Uryasev, "Optimization of conditional value-at-risk," *Journal of Risk*, vol. 2, pp. 21–42, 2000

Further requirements of Energinet

LER requirement of Energinet

Source: Energinet report [link]

"There are additional requirements for units and portfolios with limited energy reservoir (LER) units, such as batteries."

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Example: LER requirement for FCR-D Up in DK2:

"If you wish to prequalify 1 MW for FCR-D upwards, you must reserve 0.2 MW in the downwards direction for Normal State Energy Management (NEM) as well as 20 minutes of full FCR-D upwards delivery, or 0.33 MWh of energy."

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How do you interpret this requirement?

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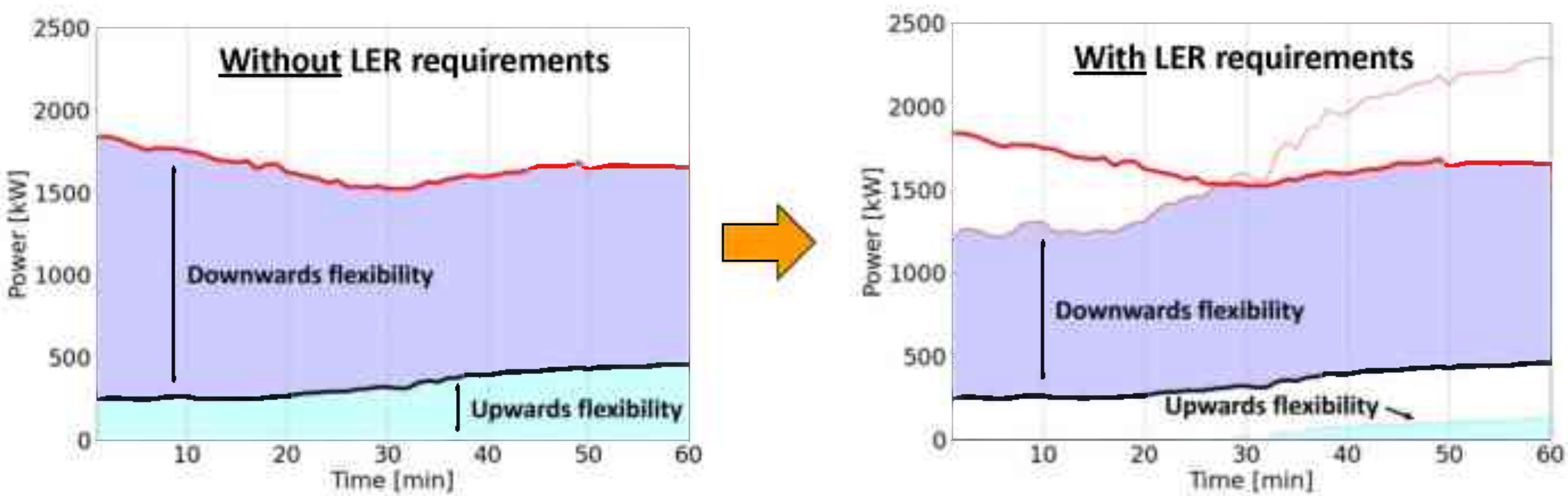
Consumption level for a random historical hour (an aggregation of 1400 EV charging boxes).



Revisited chance-constrained program with the LER requirement

For each hour *h*:

$$\underset{c_h^{\downarrow}\geq 0,\ c_h^{\uparrow}\geq 0}{\text{Maximize}} \quad c_h^{\downarrow} + c_h^{\uparrow}$$

s.t.

$$\mathbb{P}\begin{pmatrix} \frac{1}{5}c_h^{\downarrow} + c_h^{\uparrow} \leq F_{m,h}^{\uparrow} & \forall m \\ c_h^{\downarrow} \leq F_{m,h}^{\downarrow} & \forall m \\ c_h^{\downarrow} \leq F_{m,h}^{\mathrm{E}} & \forall m \end{pmatrix} \geq 1-\epsilon \qquad \forall h$$

Revisited chance-constrained program with the LER requirement

For each hour h:

FCR-D Down bid [kW] — $c_h^{\downarrow}$; FCR-D Up bid [kW] — $c_h^{\uparrow}$

$$\underset{c_h^{\downarrow}\geq 0,\ c_h^{\uparrow}\geq 0}{\text{Maximize}} \quad c_h^{\downarrow} + c_h^{\uparrow}$$

s.t.

$$\mathbb{P}\begin{pmatrix} \frac{1}{5}c_h^{\downarrow} + c_h^{\uparrow} \leq F_{m,h}^{\uparrow} & \forall m \\ c_h^{\downarrow} \leq F_{m,h}^{\downarrow} & \forall m \\ c_h^{\downarrow} \leq F_{m,h}^{\mathrm{E}} & \forall m \end{pmatrix} \geq 1-\epsilon \quad \forall h$$

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Probability distribution of upward/downward flexibility availability in minute *m*

Revisited chance-constrained program with the LER requirement

For each hour *h*:

$$\underset{c_h^{\downarrow}\geq 0,\ c_h^{\uparrow}\geq 0}{\text{Maximize}} \quad c_h^{\downarrow} + c_h^{\uparrow}$$

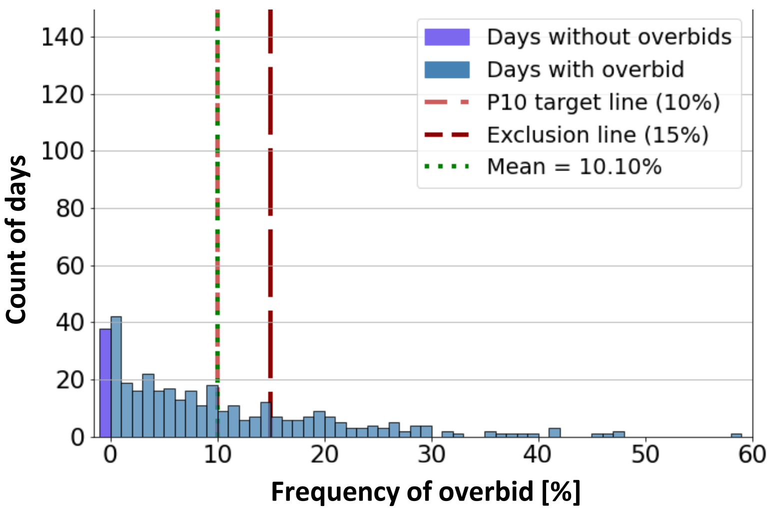
s.t.

$$\mathbb{P}\begin{pmatrix} \frac{1}{5}c_h^{\downarrow} + c_h^{\uparrow} \leq F_{m,h}^{\uparrow} & \forall m \\ c_h^{\downarrow} \leq F_{m,h}^{\downarrow} & \forall m \\ c_h^{\downarrow} \leq F_{m,h}^{\mathrm{E}} & \forall m \end{pmatrix} \geq 1-\epsilon \quad \forall h$$

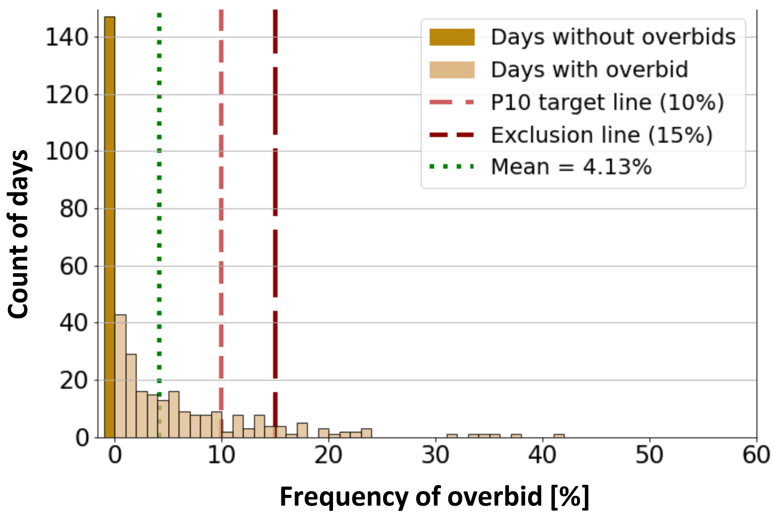
Probability distribution of downward flexibility availability in minute *m* such that aggregated battery can be charged, if the service fully activated, for the next 20 minutes

Out-of-sample results over a year

ALSO-X

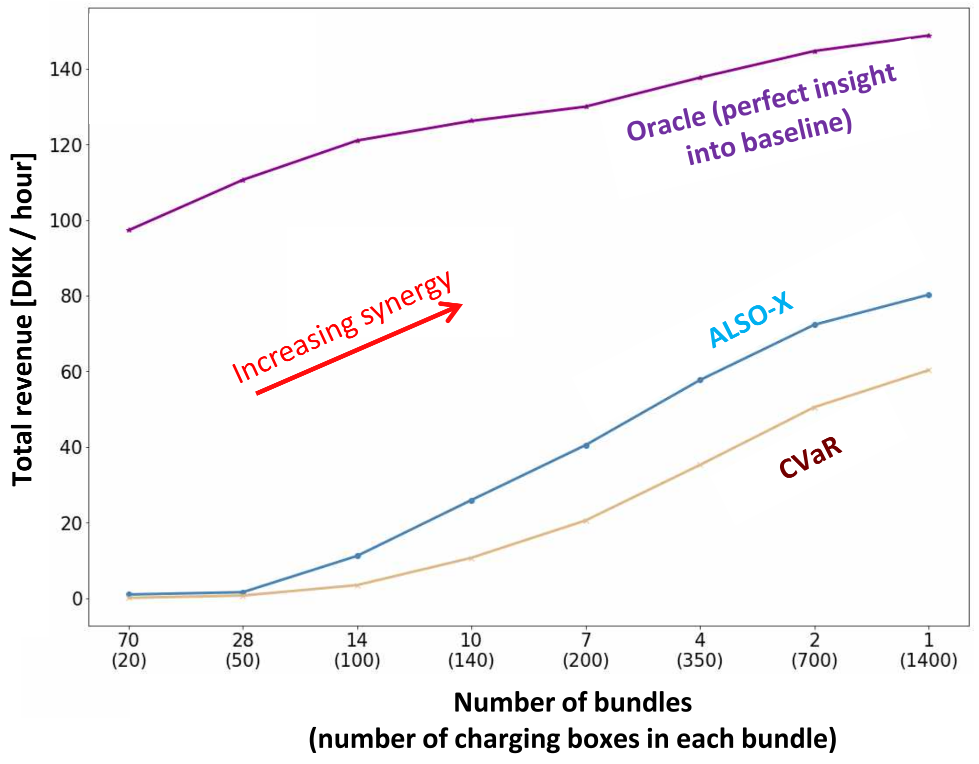


CVaR



Out-of-sample results over a year

Total profit (median) of 1400 charging boxes per hour:



Towards distributional robustness

Wasserstein distributionally robust joint chance-constrained optimization (uncertainty in the right-hand side):

$$\underset{c_h^{\downarrow}\geq 0,\ c_h^{\uparrow}\geq 0}{\text{Maximize}} \quad c_h^{\downarrow} + c_h^{\uparrow}$$

s.t.

$$\min_{\mathbb{P}\in\mathcal{P}} \ \mathbb{P}\begin{pmatrix} \frac{1}{5}c_h^{\downarrow} + c_h^{\uparrow} \leq F_{m,h}^{\uparrow} & \forall m \\ c_h^{\downarrow} \leq F_{m,h}^{\downarrow} & \forall m \\ c_h^{\downarrow} \leq F_{m,h}^{\mathrm{E}} & \forall m \end{pmatrix} \geq 1-\epsilon \qquad \forall h$$

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where

$$\mathcal{P} = \left\{ \mathbb{P} : d_W(\mathbb{P}, \hat{\mathbb{P}}_N) \leq \theta \right\}.$$

Radius (given)

Wasserstein distance

Empirical distribution

Towards distributional robustness

We adopt Proposition 2 of [5] for an exact reformulation of the joint chance constraint:

PROPOSITION 2. *For the safety set* $\mathcal{S}(\boldsymbol{x}) = \{\boldsymbol{\xi} \in \mathbb{R}^K \mid \boldsymbol{a}_m^\top \boldsymbol{x} < \boldsymbol{b}_m^\top \boldsymbol{\xi} + b_m \ \forall m \in [M]\}$, *where* $\boldsymbol{b}_m \neq \mathbf{0}$ *for all* $m \in [M]$, *the chance constrained program* (2) *is equivalent to the mixed integer conic program*

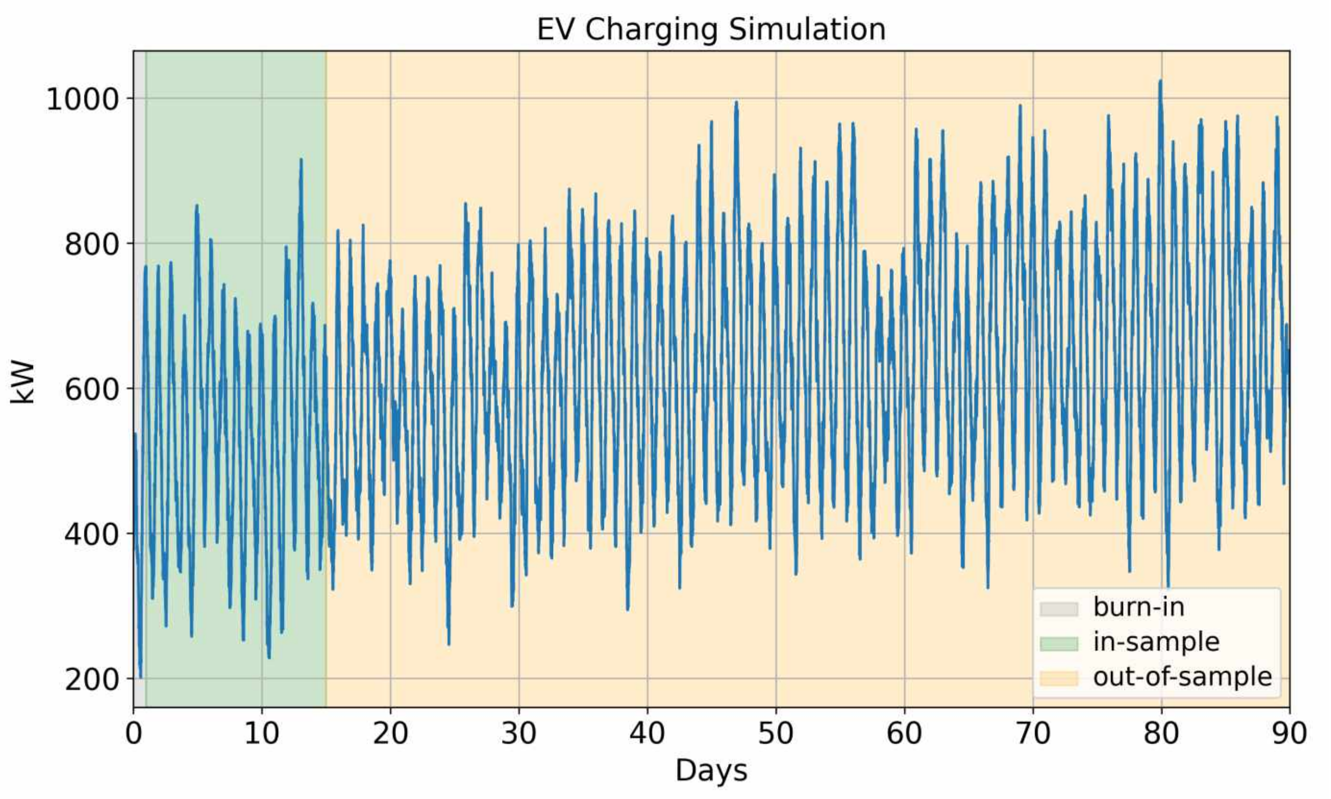
$$
\begin{aligned}
Z^\star_{\mathrm{JCC}} = \min_{\boldsymbol{q},\boldsymbol{s},t,\boldsymbol{x}} \ & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} \ & \varepsilon N t - \boldsymbol{e}^\top \boldsymbol{s} \geq \theta N \\
& \frac{\boldsymbol{b}_m^\top \hat{\boldsymbol{\xi}}_i + b_m - \boldsymbol{a}_m^\top \boldsymbol{x}}{\|\boldsymbol{b}_m\|_*} + \mathrm{M} q_i \geq t - s_i \quad \forall m \in [M], \ i \in [N] \\
& \mathrm{M}(1 - q_i) \geq t - s_i \quad \forall i \in [N] \\
& \boldsymbol{q} \in \{0,1\}^N, \ \boldsymbol{s} \geq \mathbf{0}, \ \boldsymbol{x} \in \mathcal{X},
\end{aligned}
$$

where M *is a suitably large (but finite) positive constant.*

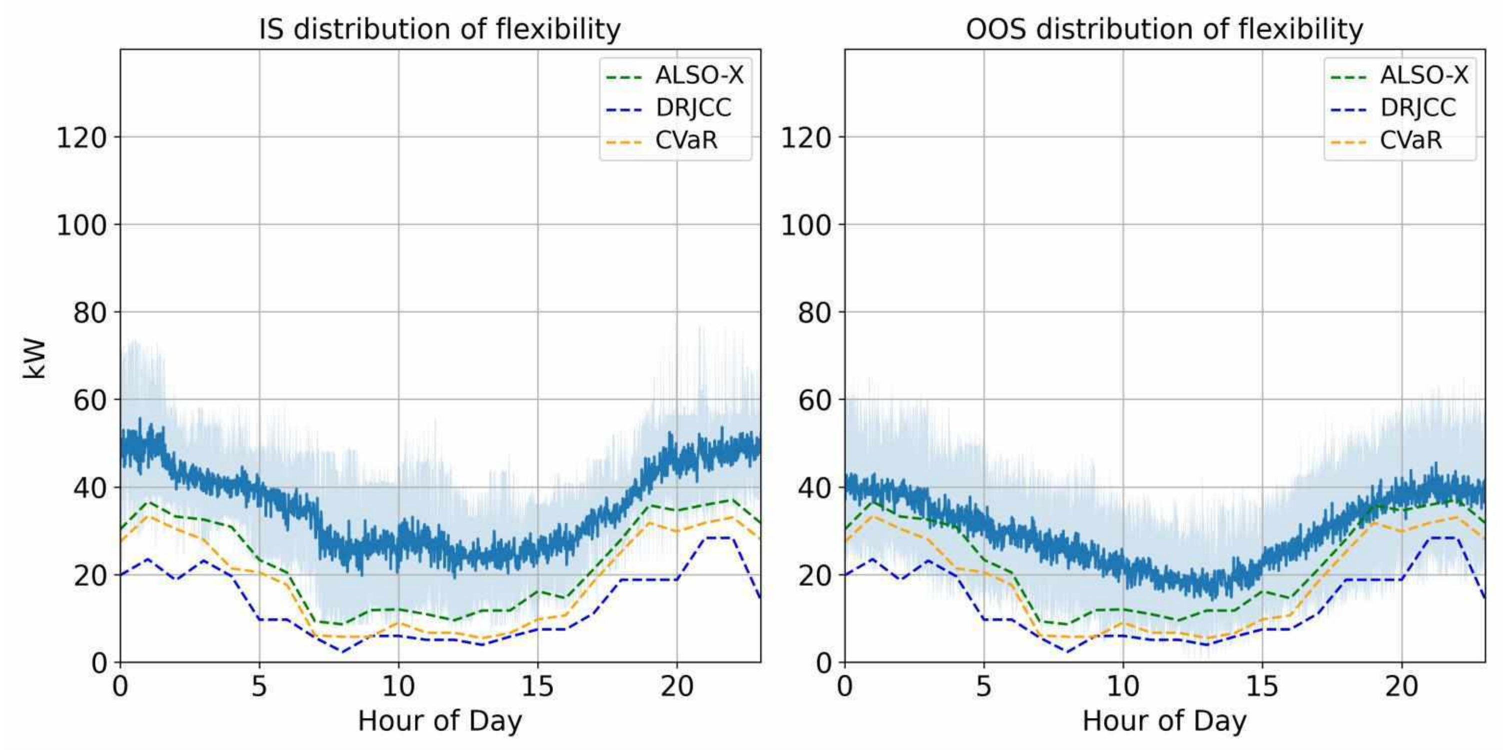
This results in a mixed-integer conic (or linear, depending on the norm) program.

[5] Z. Chen, D. Kuhn, and W. Wiesemann, "Data-driven chance constrained programs over Wasserstein balls," *Operations Research*, vol. 72, no. 1, pp. 410–424, 2024

Input data: In-sample vs out-of-sample

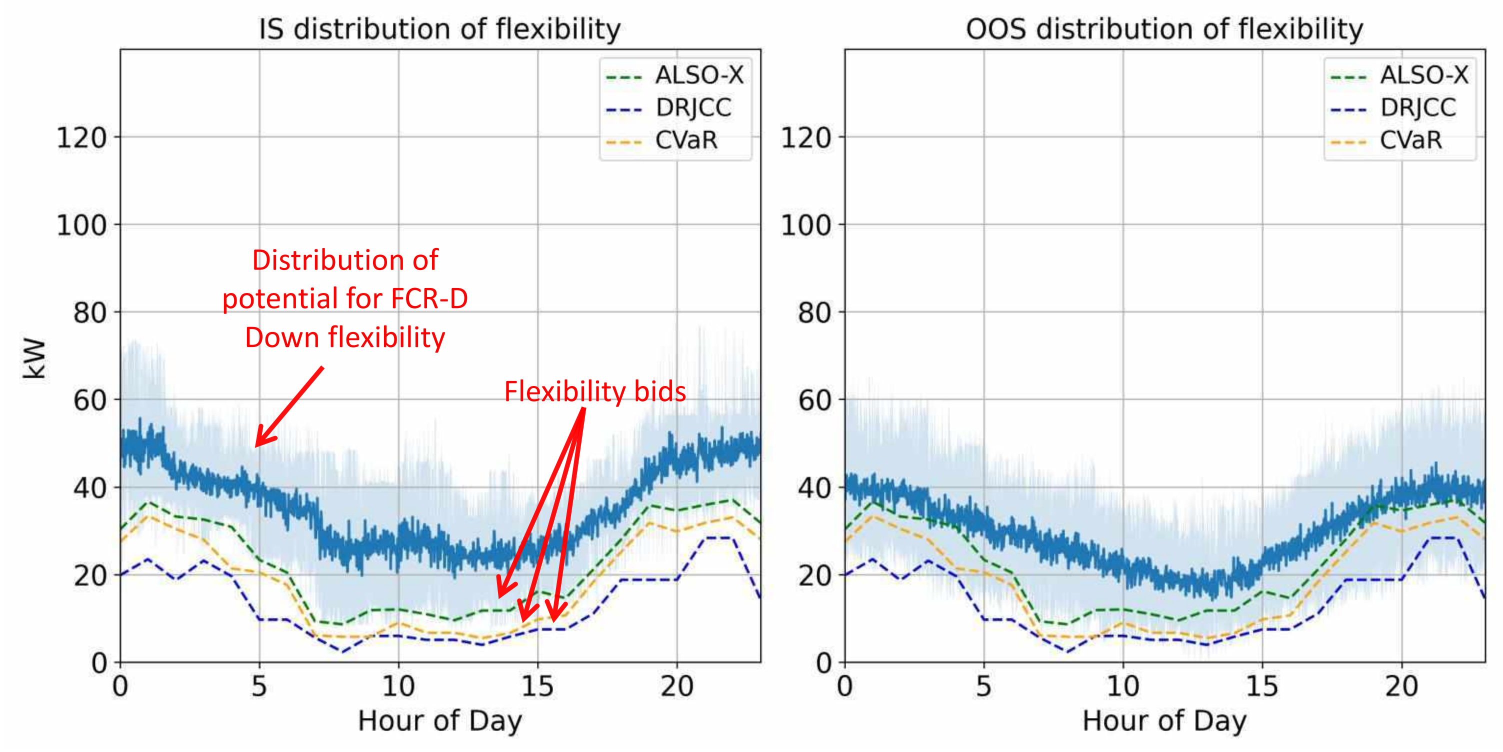


Results



IS: in-sample
OOS: out-of-sample

Results



IS: in-sample
OOS: out-of-sample

Some takeaways and potential future directions

- TSO requirements (both P90 and LER) can be modeled as a joint chance-constrained program.
- ALSO-X provides a good approximation of the chance constraint.
- CVaR is a conservative approach for solving a joint chance-constrained program.
- There is a synergy effect with more charging boxes in a bundle.

Potential future directions:

- ❑ Forecasting the baseline instead of using historical data for sampling (will it be useful?)
- ❑ Higher resolution data (enforcing constraints, e.g., per second, instead of minutes)
- ❑ Multi-market bidding (FCR-D, FCR-N, aFRR, FFR, etc)
- ❑ Does location of assets matter in low-inertia grids for frequency services?
- ❑ More heterogenous aggregation of stochastic assets (EVs + heat pumps + ...)

Further reading

G. Lunde, E. Damm, P. A. V. Gade, and JK, "Aggregator of electric vehicles bidding in Nordic FCR-D markets: A chance-constrained program," https://arxiv.org/abs/2404.12818

P. A. V. Gade, H. Bindner, and JK, "Leveraging P90 requirement: Flexible resources bidding in Nordic ancillary service markets," https://arxiv.org/abs/2404.12807

Thank you!

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